NOTES ON MULTIVARIABLE CALCULUS

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Abstract. These notes are a short review of MATH 211 and recap the key theorems, definitions, formulas, and techniques used in problem solving. Don't expect the definitions to be too rigorous, this is mainly a manual to refresh your knowledge of multivariable calculus.

1. Co-ordinate geometry and vector functions

1.1. Vectors, lines, and planes.

Definition 1. The magnitude (or norm) of a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Technique 1. The unit vector of $\mathbf{x} \in \mathbb{R}^3$ is simply the vector of magnitude 1 pointing in the direction of x:

$$\mathbf{u} = \frac{\mathbf{x}}{|\mathbf{x}|}$$

Definition 2. The dot product between vectors $\mathbf{x} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is defined as

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$|\mathbf{x} \cdot \mathbf{y}| = |\mathbf{x}||\mathbf{y}| \cos\theta$ where θ is the angle between the unit vectors of $\mathbf{x}$ and $\mathbf{y}$

Technique 2. Use dot product to find angle between vectors. We can also use the dot product to project $\mathbf{u}$ onto $\mathbf{v}$:

$$\mathrm{Proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}|^2} \mathbf{v}$$

Definition 3. The cross product between vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Also, $|\mathbf{v} \times \mathbf{u}| = |\mathbf{v}||\mathbf{u}| \sin\theta$.
Thus, $|\mathbf{v} \times \mathbf{u}|^2 + |\mathbf{v} \cdot \mathbf{y}|^2 = |\mathbf{v}|^2 |\mathbf{u}|^2$

Technique 3. The area of a parallelogram formed by two vectors $\mathbf{v}$ and $\mathbf{u}$ is $|\mathbf{v} \times \mathbf{u}|$. The volume of a parallelpiped formed by $\mathbf{v}, \mathbf{u}$ and $\mathbf{w}$ is $|\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})|$.

Technique 4. The dot product commutes but the cross product doesn't (it is anti-symmetric).

Definition 4. Given a direction vector $\mathbf{r}$ and a point $\mathbf{x_0}$, a line l has the following equation in the vector form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

In the cartesion form it is given by:

$$\frac{x - x_0}{r_1} = \frac{y - y_0}{r_2} = \frac{z - z_0}{r_3} = t$$

Technique 5. To calculate the equation of a line, find the direction vector and a point on the line.

Definition 5. Given a normal vector $\mathbf{r} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and a point $\mathbf{x_0} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, the equation of a plane Π

$$ax + by + cz = d$$

where $d = \mathbf{r} \cdot \mathbf{x_0}$

Technique 6. To calculate the equation of a plane, focus on finding the correct normal vector.

1.2. Curves and Surfaces.

1.2.1. *Curves.*

Definition 6. A curve $C \in \mathbb{R}^3$ is said to be parametrized if $\forall\, \mathbf{v} \in C$

$$\mathbf{v} = \mathbf{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$$

where f, g, and h are real functions defined on the same domain.

Definition 7. The tangent vector to C at t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The normal vector to C at t is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

The curvature of C at t is

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{T}(t)|}$$

The binormal vector to C at t is

$$\mathbf{B} = \mathbf{N} \times \mathbf{T}$$

The osculating plane is the plane formed by $\mathbf{T}$ and $\mathbf{N}$. The normal plane is formed by $\mathbf{B}$ and $\mathbf{N}$.

Result 1. $|\mathbf{r}(t)| = c \implies \mathbf{r}'(t) \perp \mathbf{r}(t)$

Definition 8. Length of a curve from $t = a$ to $t = b$ is

$$s = \int_a^b |\mathbf{r}'(t)|\, dt$$

Technique 7. How to reparametrize a curve with respect to arc length:

(1) Calculate $s = \int_0^t |\mathbf{r}'(x)|\, dx = g(t)$
(2) Calculate $t = g^{-1}(s)$.
(3) Plug in $g^{-1}(s)$ in place of t in each component.

1.2.2. *Surfaces.*

Definition 9. A surface is a set $S \subseteq \mathbb{R}^3$ is parametrized with respect to two variables, i.e

$$\mathbf{r}(u,v) = \begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix}$$

Definition 10. The two tangent vectors to a surface at (a,b) are

$$\mathbf{r}_u = \begin{bmatrix} f_u \\ g_u \\ h_u \end{bmatrix} \text{ and } \mathbf{r}_v = \begin{bmatrix} f_v \\ g_v \\ h_v \end{bmatrix}$$

$\mathbf{r}_u$ and $\mathbf{r}_v$ form the tangent plane at (a,b). The corresponding normal vector is given by $\mathbf{r}_u \times \mathbf{r}_v$

Definition 11. If we want to find the length of a curve from $t = a$ to $t = b$ then we integrate the magnitude of the tangent vector from a to b. Similarly, if we want to find the surface area of a surface on a domain D then we integrate the magnitude of the tangent plane on D, i.e:

$$A = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

1.3. **Conic sections.**

Definition 12. Given a directrix l and a focus F the eccentricity of a conic section is

$$e = \frac{|PF|}{|Pl|}$$

Definition 13. The general formula of a conic in polar co-ordinates with directrix at $x = d$ is

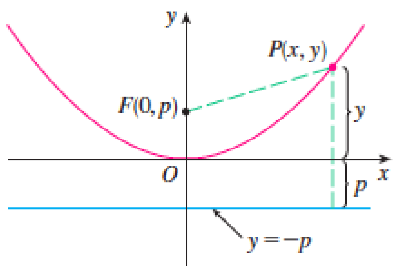
$$r(\theta) = \frac{ed}{1 + e\cos\theta}$$

Parabola

- Equation:

$$y = 4px^2$$

where p is the focus of the parabola on the y-axis
- $e = 1$
- Set of points P such that $|PF| = |Pl|$ where l is the directrix.
- Remember that the directrix is the line parallel to the tangent at the vertex passing through the reflection of the focus across the tangent at the vertex.
- Remember that while computing the focus, if you get it into the standard form like the equation above, the coefficient on the squared term is $4p$ (assuming everything else is standardised).

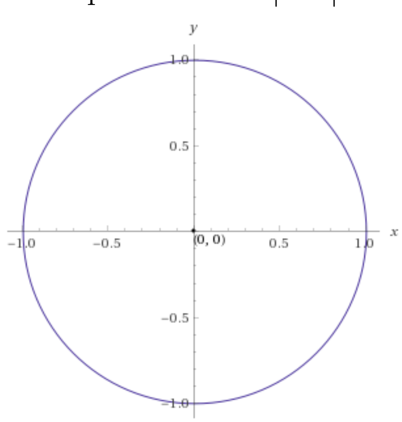


Circle

- Equation:

$$x^2 + y^2 = r^2$$

- $e = 0$
- Set of points P s.t. $|PC| = r$

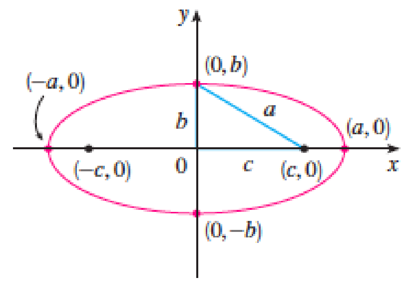


Ellipse

- Equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- $c^2 = a^2 - b^2$, where a is the major axis, b the minor axis and c the distance from centre to focus.
- $0 < e < 1$; decreasing e decreases focal length.
- Set of points P such that $|PF_1| + |PF_2| = k$

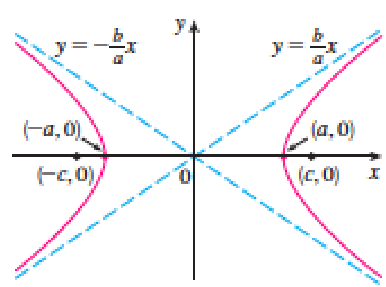


Hyperbola

- Equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

- $c^2 = a^2 + b^2$, where a is the major axis, b the minor axis and c the distance from centre to focus.
- $e > 1$; increasing e decreases focal length.
- Set of points P such that $||PF_1| - |PF_2|| = k$



The above illustration of conic sections is only restricted to conic sections in the standard form (i.e centered at the origin). If you encounter either a translated conic or a turned conic, rotate (use $\theta + \alpha$ in the polar form) and translate it back into standard position, compute whatever you need to compute and remember to translate and rotate any points you find back to the original position.

1.4. **Quadric Surfaces.**

Technique 8. The main question around quadric surfaces is that you need to sketch the surface given an equation. The way to do it is to fix each x, y, and z co-ordinate and check which conic sections are formed in each case.

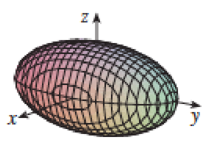
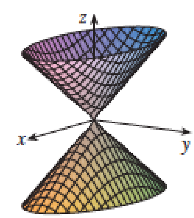
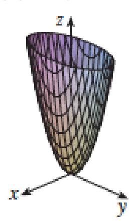
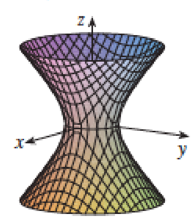
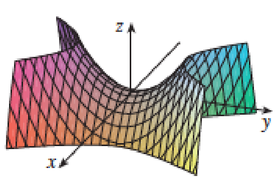
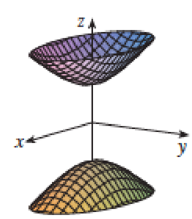
Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

FIGURE 1. Graphs of quadric surfaces

Finally, to convert any planar curve $f(x_i, x_j) = 0$ into a surface which is formed by the rotation of the curve around one of the axes, do the following: Pick the axis you want to rotate around, say x_i. In the curve equation, replace the other variable x_j by $\pm\sqrt{x_j^2 + x_k^2}$. For example, suppose you want to make a cone from the lines $x^2 - z^2 = 0$ around the z axis. Then replace x with $\pm\sqrt{x^2 + y^2}$ to get $x^2 + y^2 - z^2 = 0$. This is the equation of a cone!

2. Differential Multivariable Calculus

Definition 14. The limit of a multivariable function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ at (a, b) is c iff

$$\forall \epsilon > 0, \exists \delta \text{ s.t. } ||(x, y) - (a, b)|| < \delta \implies ||f(x, y) - z|| < \epsilon$$

This is denoted as $\lim_{(x,y)\to(a,b)} f(x, y) = z$.

Technique 9. Sometimes you need to calculate a δ as a minimum of two numbers, depending on ϵ. In this case, it helps starting with the case $\epsilon \geq 1$ followed by $\epsilon < 1$.

Technique 10. To prove that f is not continuous at (x, y), show that the limit on two different paths is different. Typically the paths used are the $y = y_0$ line, $x = x_0$ line or the line $y - y_0 = m(x - x_0)$.

Definition 15. The partial derivative of f with respect to x and y is:

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \text{ and } f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

f is said to be differentiable at (x, y) iff f_x and f_y exist in some open ball around (x, y) and are continuous at (x, y).

Technique 11. If f is differentiable, then you calculate partial derivatives with respect to a variable by doing one-variable differnentiation with respect to that variable and treating all the other variables as constant. If f is not differentiable and the partial derivatives exist, then you must use the limit definition.

Thus, before doing any partial differentiation, always check for differentiability!

Technique 12. Clairaut's theorem states that if f has continuous second partial derivatives at (x, y) then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Definition 16. Given a differentiable function f, the tangent plane to the surface $\begin{bmatrix} x \\ y \\ f(x, y) \end{bmatrix}$ at $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ where $z_0 = f(x_0, y_0)$ has the equation

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Note that this is also the first order taylor approximation to f at (x_0, y_0)

Definition 17. The differential dz of a differentiable function $z = f(x, y)$ is

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Technique 13. You could be asked to compute Δz and dz. dz is given by the above formula and Δz is just computed by $f(x + \Delta x, y + \Delta y) - f(x, y)$. You will typically be provided the increments in x and y.

Definition 18. The directional derivative to f at (x, y) in the direction $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ is

$$D_{\mathbf{u}} f = \lim_{h \to 0} \frac{f(x + ah, y + bh) - f(x, y)}{h}$$

Definition 19. Note that the ∇ operator is the "vector"

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix}$$

The gradient is defined as ∇f

For a two variable function, the third component in the gradient vector is 1. (Because $z = f$ so $\frac{\partial f}{\partial z} = 1$)

Technique 14. Note that when f is differentiable then $D_{\mathbf{u}}(f) = \nabla f \cdot \mathbf{u}$. THIS IS IMPORTANT! A common question is to compute a directional derivative from the limit definition! Once again, when f isn't differentiable, you need to use the limit definition.

Technique 15. $\nabla f(a, b)$ is perpendicular to the tangent plane of the surface $\begin{bmatrix} x \\ y \\ f(x, y) \end{bmatrix}$ at $\begin{bmatrix} a \\ b \\ f(a, b) \end{bmatrix}$. Thus to compute the tangent plane to the surface defined by a function, compute ∇f which is the normal vector. Then use the normal vector definition of a plane to find the equation.

Technique 16. The multivariable chain rule when $z = f(x(t), y(t))$ is

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

The generalized chain rule when $z = f(x_1(t_1, \ldots, t_n), \ldots, x_m(t_1, \ldots, t_n))$ is

$$\frac{\partial z}{\partial t_i} = \sum_{k=1}^{m} \frac{\partial z}{\partial x_k}\frac{\partial x_k}{\partial t_i}$$

Definition 20. The determinant D of the Hessian matrix $\mathcal{H}$ $\mathcal{H}$ of f is

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy} f_{yx} = f_{xx} f_{yy} - (f_{xy})^2 \text{ when } f \text{ is twice differentiable}$$

Definition 21. The Lagrangian L of an objective function f and its constraint $g(x, y) = 0$ is

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

At the critical points the first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \quad \frac{\partial \mathcal{L}}{\partial y} = 0 \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

Technique 17. There are two kinds of multivariable optimization problems:

(1) Optimize over a region
To optimize over a (simply connected) region, follow these steps:
- Check if the region is closed or open. If closed then check the value of f on the boundary of the region. If open, go to the next step.
- Solve $f_x = 0$ and $f_y = 0$ on the open region and find the critical points.
- If (a, b) is a critical point then
  (a) (a, b) is a local maximum $\iff D > 0$ and $f_{xx} < 0$
  (b) (a, b) is a local minimum $\iff D > 0$ and $f_{xx} > 0$
  (c) (a, b) is a saddle point $\iff D < 0$
- If the region was open, you are done classifying the critical points. If it was closed, compare the critical points to the values found on the boundary.

(2) Optimize with respect to a constraint
To optimize $f(x, y)$ with respect to $h(x, y) = k$, follow these steps:
- Let $g(x, y) = h(x, y) - k = 0$.
- Compute $\mathcal{L}$ and solve for the first order conditions.
- Equivalently, you need to solve the following equations:

$$\nabla f = \lambda \nabla h$$
$$h(x, y) = k$$

Adjust the first order conditions on the Lagrangian when there are more than two variables or more than one constraint. For multiple constraints, simply subtract a multiplier times the constraint from $\mathcal{L}$ and take the first-order condition with respect to that multiplier.

3. Integral Multivariable Calculus

Definition 22. Let $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^m$ where

$$\mathbf{f}\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$$

Then the Jacobian $\frac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_n)}$ of f is the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

The Jacobian determinant is the determinant of the Jacobian matrix, i.e

$$D = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{vmatrix}$$

Technique 18. Four common double integrals:

(1) Over a cartesian product
If $D = [a,b] \times [c,d]$ then

$$\iint_D f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

If f is continuous on D then according to Fubini's theorem,

$$\iint_D f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

We integrate the inner integral by doing single variable integration, keeping the other variable constant.

(2) Over a general region bounded by curves $x = a, x = b, y = g_1(x), y = g_2(x)$

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Note that the case when x is given in terms of y is the same, you just switch the order of integration.

(3) Over a region that looks better in terms of polar co-ordinates:
Here we use the multivariable substitution rule where we convert D through a bijection $\mathbf{f} : \mathbb{R}^+ \times [0, 2\pi) \mapsto \mathbb{R}^2$ where $\mathbf{f}\Big(\begin{bmatrix} r \\ \theta \end{bmatrix}\Big) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$. Hopefully, $\mathbf{f}^{-1}(D) = [r_1, r_2] \times [\theta_1, \theta_2]$

$$\iint_D f(x,y)\,dA = \iint_{\mathbf{f}^{-1}(D)} f(r\cos\theta, r\sin\theta)\left|\frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_n)}\right| dS = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta)\,r d\theta\,dr$$

(4) The surface area of a surface formed by $z = f(x,y)$ over the domain D is

$$\iint_D \sqrt{1 + \big(\frac{\partial f}{\partial x}\big)^2 + \big(\frac{\partial f}{\partial y}\big)^2}\,dA$$

This is actually a specific case of the formula for surface area of a general parametric surface.

Remember! The general skill required for integrating is to be able to conceive the domain well. Whenever you can, draw a picture and figure out the bounding curves of the region. If it does not naturally occur to you how you can simplify the integral, think about the above three methods.

Technique 19. Four common triple integrals:

(1) Integration over a cartesian product:

$$\iiint_D f(x,y,z)\,dV = \int_a^b \int_c^d \int_s^t f(x,y,z)\,dz\,dy\,dx$$

(2) Integration over a solid bounded by curves:

$$\iiint_D f(x,y,z)\,dV = \int_a^b \int_{h_1(x)}^{h_2(y)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

when z is bounded between two surfaces, y is bounded between two curves, x is bounded between two points.

(3) The Cylindrical transformation: $\mathbf{f}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}$. Then the Jacobian determinant is r and typically, $\mathbf{f}^{-1}(D) = [z_1, z_2] \times [\theta_1, \theta_2] \times [r_1, r_2]$.

$$\iiint_D f(x,y,z)\, dV = \int_{z_1}^{z_2} \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta, z)\, r d\theta\, dr\, dz$$

(4) The spherical transformation: $\mathbf{f}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{bmatrix}$ where θ is the angle of the vector in the $x-y$ plane and ϕ the angle of the vector with the z axis. In this case the Jacobian determinant is $r^2\sin\phi$, the inverse image is typically a cartesian product and the integral is

$$\iiint_D f(x,y,z)\, dV = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} f(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)\, r^2 \sin\phi\, d\phi\, d\theta\, dr$$

Once again, the most important thing is to draw the region to figure out the integration domain.

4. Vector Calculus

Definition 23. A vector field is a map $\mathbf{F} : \mathbb{R}^n \mapsto \mathbb{R}^n$. The curl and divergence of $\mathbf{F}$–when $\mathbf{F}$ is in two or three dimensions–are given as follows:

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} \text{ and } \operatorname{Div} \mathbf{F} = \nabla \cdot \mathbf{F}$$

In other words,

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{bmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ -(\frac{\partial f_3}{\partial x} + \frac{\partial f_1}{\partial z}) \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{bmatrix}$$

and

$$\operatorname{Div} \mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Definition 24. A conservative vector field F is one for which $\exists\, f : \mathbb{R}^3 \mapsto \mathbb{R}$ s.t.

$$\mathbf{F} = \nabla f$$

Technique 20. There are four common ways of computing a line integral:

A line integral of a scalar field $f : \mathbb{R}^3 \mapsto \mathbb{R}$ on a curve C parametrized as $\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ is calculated using the following equality:

$$\int_C f\,d\mathbf{r} = \int_a^b f((x(t), y(t), z(t))|\mathbf{r}'(t)|\,dt$$

A line integral of a vector field $F : \mathbb{R}^3 \mapsto \mathbb{R}^3$ on a curve C is calculated using the following equality:

$$\int_C \mathbf{F} \cdot\, d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(\mathbf{t})) \cdot \mathbf{r}'(t)\,dt$$

The line integral can also be sometimes denoted as

$$\int_C \mathbf{F} \cdot\, d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$$

where

$$\int_C P\,dx = \int_a^b P(x(t), y(t), z(t))x'(t)\,dt$$

If $\mathbf{F}$ is conservative where $\mathbf{F} = \nabla f$ and the curve is parametrized between $t = b$ and $t = a$ then from the Fundamental theorem of calculus,

$$\int_C \mathbf{F} \cdot\, d\mathbf{r} = f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))$$

If C is a closed curve on the plane and its orientation is anti-clockwise (i.e its parametrization takes one in the anti-clockwise direction) then from Green's theorem,

$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

There is one technique used for computing line integrals of a special kind, those of general closed curves. This technique uses Stokes' theorem.

Technique 21. There are 5 common surface integrals. Here's how to solve them.

(1) Integrating a scalar field $f : \mathbb{R}^3 \mapsto R$ on a surface S with D as the domain of parametrization:

$$\iint_S f\,d\mathbf{S} = \iint_D f(x(u, v), y(u, v), z(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

(2) Integrating a vector field $\mathbf{F} : \mathbb{R}^3 \mapsto \mathbb{R}^3$:

$$\iint_S \mathbf{F}\,d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\,dA$$

(3) According to Stokes' theorem, if C is a curve bounding the surface S such that the positive orientation of S induces a positive orientation on C, i.e if you take a directed walk along C with your head pointed towards the normal vector $\mathbf{n}$ of S then S will always be to your left, then

$$\oiint_S \operatorname{curl} \mathbf{F}\,d\mathbf{S} = \oint_C \mathbf{F}\,d\mathbf{r}$$

Note that the surface needn't be closed. The following result, the divergence theorem, is used to compute the surface integral on closed surfaces:

$$\iint_S \mathbf{F}\, d\mathbf{S} = \iiint_D \operatorname{Div} \mathbf{F}\, dV$$