ON THE CURVE SHORTENING FLOW

SHASHANK SULE

1. Introduction

A curve $\gamma : S^1 \times [0,T) \mapsto \mathbb{R}^2$ where $T > 0$ evolves by the *curve shortening flow* if $\gamma_t := \gamma(\cdot, t)$ is embedded for every $t \in [0,T)$ and γ satisfies the following non-linear parabolic equation for every $(x,t) \in S^1 \times [0,T)$:

$$\partial_t \gamma(x,t) = \kappa(x,t) N(x,t) \tag{1.1}$$

Here $\kappa(x,t)$ is the curvature of γ_t at $x \in S^1$. Usually $\kappa(x,t) = \langle \partial_x T(x,t), N(x,t) \rangle$ where $N(x,t) = JT(x,t)$ (J is the $\pi/2$ counter-clockwise rotation matrix in $SO(2)$ and $T(\cdot,t)$ is the unit tangent vector of γ_t). Since γ is embedded, it can be (re)parametrized by arc length so $\kappa(x,t)N(x,t) = \partial_s^2 \gamma(s,t)$ giving a reformulation of Equation 1.1 in terms of arc-length s:

$$\partial_t \gamma(s,t) = \partial_s^2 \gamma(s,t) \tag{1.2}$$

Equation 1.2 reveals that the curve shortening equation is a heat equation for plane curves. However, we must remark with caution that the above equation is not linear with respect to x because arc-length s may depend non-linearly on x. As a result given curves $\gamma(x,t) = \gamma_1(x,t) + \gamma_2(x,t)$ it may not be true that $\partial_s \gamma = \partial_s \gamma_1 + \partial_s \gamma_2$. For example, ∂_s is not linear for a curve given by the (componentwise) addition of parabola to a cubic both parametrized by x. Another way to see the non-linearity of 1.2 is to check what happens when X is not parametrized by arc length; in that case we have that $X_s = T(x(s)) = X'(x)/|X'(x)|$. Then according to Equation 1.2, $X_{ss} = \frac{\partial}{\partial s} X_s = \frac{dT}{dx}\frac{dx}{ds} = \frac{1}{|X'(x)|}(\frac{X'(x)}{|X'(x)|})' = X_t$, which is clearly non-linear.

Before embarking on a quest to study the properties of curves which evolve by the curve shortening flow, it is worth mentioning the "weak" version of Equation 1.1:

$$\langle N, \partial_t \gamma(x,t) \rangle = \kappa(x,t) \tag{1.3}$$

Equation 1.1 implies Equation 1.3 but the converse is not true. However, Equation 1.3 is arguably more important for studying the curve shortening flow because for a fixed $x \in S^1$, the trajectory $\gamma(x,\cdot))$ can be parametrized such that it is not orthogonal to γ_t at every t without changing Equation 1.2. To see this more clearly, suppose $(x,t) \mapsto (\varphi(x,t), t)$ is a time-dependent reparametrization. Then if we have a curve $Y(x,t) = X(\varphi(x,t),t)$ and X flows by curve shortening then by the chain rule we have

$$Y_t(x,t) = \frac{1}{|Y_x(x,t)|}\frac{\partial}{\partial x}(\frac{Y_x(x,t)}{|Y_x(x,t)|}) + (\frac{\varphi_t(x,t)}{\varphi_x(x,t)})Y_x(x,t)$$

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Consequently, a time dependent reparametrization causes Y to pick up a tangential term; the parametrization can always be reversed to remove that tangential term as long as the curve evolves by curve shortening in the arc-length parameter.

Thus Equations 1.2 and 1.3 are equivalent up to reparametrization so we will study either one of the three depending on convenience.

Example 1.1. Suppose $\gamma(x,t) = r(t)(\cos(x), \sin(x))$ be a circle of radius $r(t) \geqslant 0$. Then $\gamma_x(x,t) = r(t)(-\sin(x), \cos x)$ and $|\gamma_x(x,t)| = |r(t)| = r(t)$ so $T(x,t) = \gamma_x(x,t)/|\gamma_x(x,t)| = (-\sin(x), \cos x)$. Next,

$$\begin{aligned}\frac{dT}{ds} &= \frac{dT}{dx}\frac{dx}{ds} \\ &= \frac{T_x(x,t)}{|\gamma_x(x,t)|} \\ &= \frac{1}{|r(t)|}(-\cos(x), -\sin(x)) \\ &= \frac{1}{|r(t)|}N(x,t) = \kappa(x,t)N(x,t)\end{aligned}$$

Lastly, we have that $\gamma_t(x,t) = r'(t)(\cos(x), \sin(x))$ so according to Equation 1.1 we have that

$$r'(t)(\cos(x), \sin(x)) = \kappa(x,t)N(x,t) = \frac{1}{|r(t)|}(-\cos(x), -\sin(x))$$

This implies the ODE $r'(t) = -1/r(t)$. Using separation of variables we get that

$$\int_{r(0)}^{r(t)} \rho \, dr = \int_0^t \tau \, d\tau$$

so $\frac{1}{2}((r(t))^2 - (r(0))^2) = -t$. Rearranging, we have that $r(t) = \sqrt{r(0) - 2t}$. Thus a circle with initial radius $R(0) > 0$ is always a solution to the curve shortening flow. In fact, since $r(t) = \sqrt{(r(0))^2 - 2t}$, the flow is only well-defined until time $T = (r(0))^2/2$ and the radius shrinks monotonically until γ_t collapses to a "round point" as $t \to T$.

Example 1.2. Can the translations of a graph of a twice-differentiable function be solutions to the curve shortening flow? Such a solution is called a soliton. Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ be a C^2 function where $\gamma(x,t) = (x, f(x) + ct)$. Then $\gamma_t(x,t) = (0,c)$ so $\gamma_t(x,t) = \kappa(x,t)N(x,t)$ is always $(0,1)$ which means that Γ_t is a line parallel to the x-axis. Since this answer is not very interesting, it is worth looking at the solutions to the weak problem. According to Equation 1.3, given $\gamma : S^1 \times [0,T) \mapsto \mathbb{R}^2$ we have that

$$\langle N(x,t), \gamma_t \rangle = \kappa(x,t)$$

Now, given $\gamma(x,t) = (x, f(x) + ct)$, $\gamma_x = (1, f')$ so $T(x,t) = \frac{1}{\sqrt{1+(f')^2}}(1, f')$. Then $N = \frac{1}{\sqrt{1+(f')^2}}(-f, 1)$ Moreover

$$\begin{aligned}
\frac{dT}{ds} &= \frac{1}{\sqrt{1+(f')^2}} \begin{bmatrix} -\frac{1}{1+(f')^2}\frac{2f'f''}{2\sqrt{1+(f')^2}} \\ \frac{f''\sqrt{1+(f')^2} - f'\frac{2f'f''}{2\sqrt{1+(f')^2}}}{1+(f')^2} \end{bmatrix} \\
&= \frac{f''}{(1+(f')^2)^{3/2}} \begin{bmatrix} \frac{-f'}{\sqrt{1+(f')^2}} \\ \sqrt{1+(f')^2} - \frac{(f')^2}{\sqrt{1+(f')^2}} \end{bmatrix} \\
&= \frac{f''}{(1+(f')^2)^{3/2}} \begin{bmatrix} \frac{-f'}{\sqrt{1+(f')^2}} \\ \frac{1}{\sqrt{1+(f')^2}} \end{bmatrix} \\
&= \frac{f''}{(1+(f')^2)^{3/2}} N(x,t)
\end{aligned}$$

Consequently, we get that $\kappa(x,t) = \frac{f''}{(1+(f')^2)^{3/2}}$. From Equation 1.3 we have that

$$\langle \gamma_t(x,t), N(x,t) \rangle = \frac{c}{\sqrt{1+(f')^2}} = \kappa(x,t) = \frac{f''}{(1+(f')^2)^{3/2}}$$

Cancelling $\frac{1}{\sqrt{(1+(f')^2)}}$ on both sides, we once again obtain an ODE:

$$c = \frac{f''}{1+(f')^2} = (\arctan(f'))'$$

Consequently, we have that $\arctan(f') = cx+d$. Rearranging we get the ODE $f' = \tan(cx+d)$ which, when given the condition $f(x_0)$ has the solution $f(x) = -\frac{1}{c}\ln(\cos(cx+d)) + b$. Now, given this form, f is well-defined whenever $-\frac{1}{c}\ln(\cos(cx+d))$ is well-defined; consequently, we restrict $cx + d \in (-\pi/2, \pi/2)$. The *Grim Reaper Curve* is γ when $d = b = 0$ and $c = 1$ well defined for $x = (-\pi/2, \pi/2)$. In this case, Γ_t evolves by vertical translations.

Example 1.3. In Example 1.2 the Grim Reaper was derived as a particular solution to $c = (\arctan(f'))'$ for $c = 1$ and $d = b = 0$. Consequently, an entire family of curves evolving by the curve shortening flow can be obtained by setting $d, b \in \mathbb{R}$ and $c \neq 0$. Moreover, the parameter b only controls the vertical position of the curve in the plane so it can be assumed that $b = 0$ (otherwise, we can translate the y axis by $-b$ without changing the x axis). We denote $\gamma^{c,d}(x,t) = (x, -\frac{1}{c}\ln\cos(cx+d)+ct) = (x, -\frac{1}{c}\ln\cos(M_{c,d}(x)))+ct$ where $M_{c,d} = cx+d$ is an isometry of $\mathbb{R}$. The solution $\gamma^{c,d}$ still evolves by translation, but its orientation depends on the parameters c and d. As a result, the Grim Reaper is a *self-similar solution* to the curve shortening flow: it is invariant of orientation and flows by translation. One more way to see the self-similarity of the Grim Reaper is to consider the implicit formulation of $\gamma^{1,0}$ as $\gamma^{1,0} : S^1 \times \mathbb{R}$ where $\gamma(s,t) = (x(s,t), y(s,t))$ satisfying $e^{-y} = e^{-t}\cos(x)$. If we assume that y is once differentiable with respect to x, then we get the Grim Reaper as a solution by differentiating the functional equation and assuming $x_t(s,t) = 0$. Note that we may tweak the functional equation to any arbitrary isometry on y and x. Then a translating Grim Reaper is the solution to $e^{-(ay+b)} = e^{-t}\cos(cx+d)$ with the explicit solution given by $y = -\frac{1}{a}\ln c\cos(cx+d) + \frac{t-b}{a}$. Dilating the x-axis by $1/c$ and the y axis by a and translating by b we get $y = \ln c\cos(x + d) + t$, resolving it in the two-parameter form obtained by solving the ODE for various parameters of c and d. Lastly, the equations $e^{-y} = e^{-t}\cos(x)$ and $e^{y} - e^{-t}\cos(x)$ can actually be combined to get another pair of solutions, the paperclip $\cosh(y) = \cos(x)e^{-t}$ and the hairclip $\sinh(y) = \cos(x)e^{-t}$.

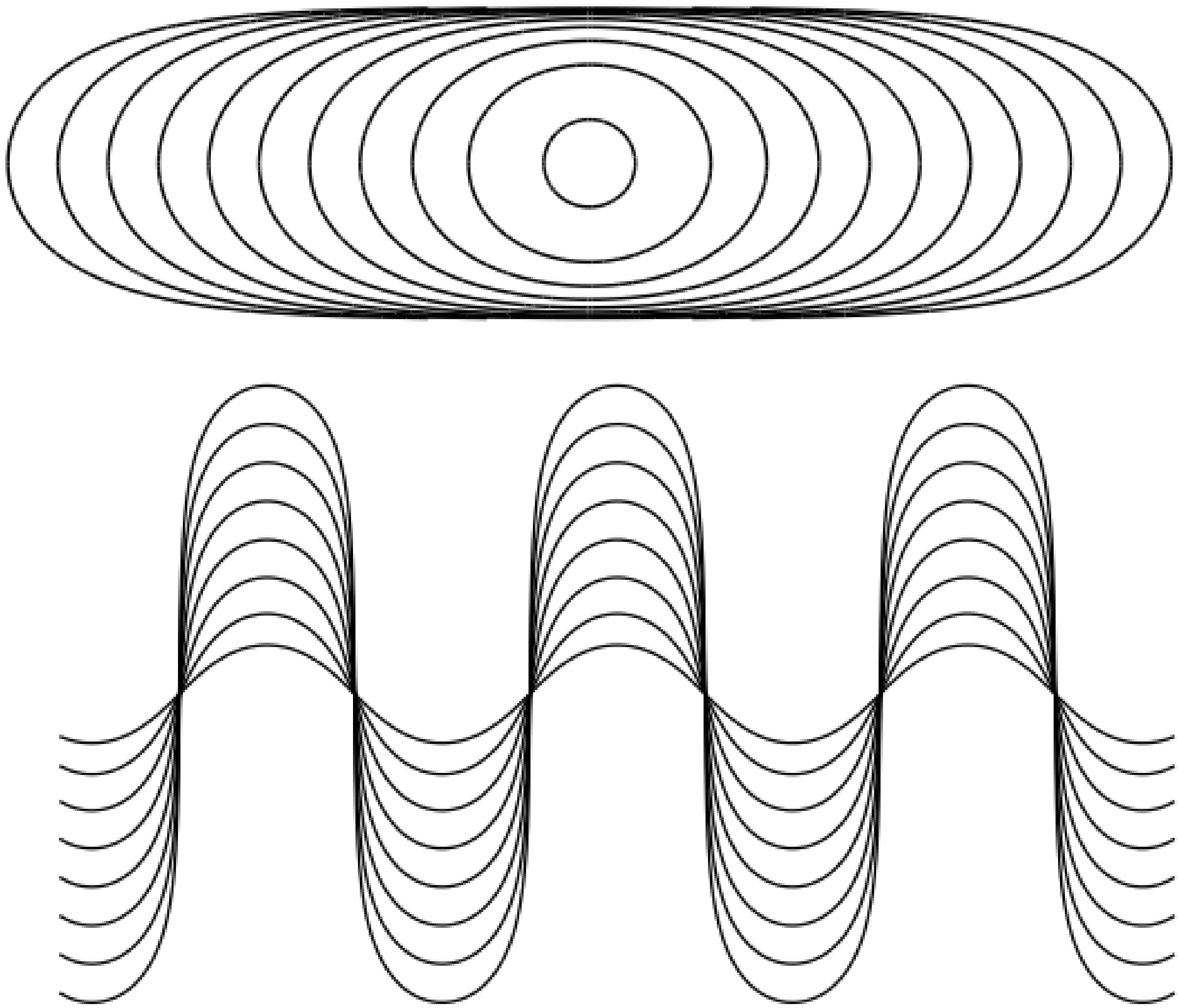

FIGURE 1. The paperclip (above) and hairclip (below) solutions to the curve shortening flow

2. Geometric Properties of the Curve Shortening Flow

In this section, we discuss some fundamental geometric properties of solutions to the curve shortening flow. Before we begin, we remark that most geometric properties of such solutions can be determined by computing the correct functional of the curve Γ_t. As a simple example, we prove that the areas and lengths of solutions to the curve shortening flow. We start by recalling the definitions of area and length; we assume $\Gamma \subset \mathbb{R}^2$ is a closed embedded curve of unit length with an arc length parametrization $\Gamma : S^1 \mapsto \mathbb{R}$ such that $\Gamma(0) = \Gamma(1)$ and $T(0) = T(1)$.

Definition 2.1. Since Γ is closed, by the Jordan Curve Theorem, it partitions the plane into an interior Σ and an exterior $\Sigma^c \cap \Gamma^c$. Then the *area* of Γ is $A(\Gamma) = \int_\Sigma dx$.

Remark 2.1. Note that $A(\Gamma) = m(\Sigma)$ where m is the Lebesgue measure on $\mathbb{R}^2$.

Definition 2.2. The *Length* of Γ is the following:

$$L(\Gamma) = \sup_{\mathcal{P}} \Big\{ \sum_{1 \leqslant i \leqslant n-1} |\Gamma(t_{i+1}) - \Gamma(t_i)| \,\Big|\, [t_i, t_{i+1}] \in \mathcal{P}, |\mathcal{P}| = n \Big\}$$

Here $\mathcal{P}$ is a partition of S^1. Note that since Γ is embedded,

$$L(\Gamma) = \int_{S^1} |\Gamma'(x)|\, dx \tag{2.1}$$

Before we begin investigating the curve shortening flow on Γ we would like to first seek a formula for area similar to that of $L(\Gamma)$, explicitly involving the parametrization of Γ. To that end, since $\partial\Sigma = \Gamma$ we may use Green's theorem:

$$\int_\Sigma dx = \frac{1}{2} \oint_\Gamma -y\, dx + x\, dy = \oint_\Gamma (-y, x) \cdot d\hat{r} = \frac{1}{2} \int_{S^1} \langle J\Gamma, T \rangle\, dx \tag{2.2}$$

With Equations 2.1 and 2.1 in hand, we can start to study how area and length change with respect to time when Γ_t flows by curve shortening. Assume for the remainder of the section that $\Gamma : S^1 \times [0, T) \mapsto \mathbb{R}^2$ where $\Gamma_t = \Gamma(S^1, t)$ is a simple closed embedded curve for every $t \in [0, 1)$ and that Γ flows by curve shortening.

Definition 2.3. The *first variations* of Area and Length are $\dfrac{dA(\Gamma_t)}{dt}$ and $\dfrac{dL(\Gamma_t)}{dt}$ respectively.

Note that when Γ_t is simple, closed, and embedded, the first variations are well-defined since Equations 2.1 and 2.1 assure that area and length are C^1 with respect to time. The following calculations reveal a connection with curvature when Γ flows by curve shortening.

Proposition 2.1. *Let Γ_t flow by curve shortening. Then*

$$\frac{d}{dt} A(\Gamma_t) = -\int_{S^1} \kappa(x, t)\, dx = -2\pi$$

Proof.

$$\begin{aligned}
\frac{d}{dt} A(t) &= \frac{1}{2} \frac{d}{dt} \int_{S^1} \langle J\Gamma, T \rangle\, dx \\
&= \frac{1}{2} \frac{d}{dt} \int_{S^1} \langle J\Gamma_t, T \rangle + \langle J\Gamma, T_t \rangle\, dx \\
&= \frac{1}{2} \int_{S^1} \langle \kappa J N, T \rangle + \langle J\Gamma, \Gamma_{xt} \rangle\, dx \\
&= \frac{1}{2} \int_{S^1} \langle -\kappa T, T \rangle - \langle J\Gamma_x, \Gamma_t \rangle\, dx \\
&= \frac{1}{2} \int_{S^1} -\kappa - \langle N, \kappa N \rangle\, dx \\
&= \frac{1}{2} \int_{S^1} -2\kappa\, dx = \int_{S^1} \kappa\, dx = -2\pi
\end{aligned}$$

To obtain the fourth line from the third, we use integration by parts. ∎

Proposition 2.2. *Let Γ_t flow by curve shortening. Then*

$$\frac{d}{dt}L(\Gamma_t) = -\int_{S^1} \kappa(x,t)^2\, dx$$

Proof.

$$\begin{aligned}
\frac{d}{dt}L(t) &= \frac{d}{dt}\int_{S^1} |\Gamma_x(x,t)|\, dx \\
&= \frac{d}{dt}\int_{S^1} \sqrt{\langle \Gamma_x, \Gamma_x\rangle}\, dx \\
&= \int_{S^1} \frac{\langle \Gamma_{xt}, \Gamma_x\rangle}{|\Gamma_x|}\, dx \\
&= \int_{S^1} \langle \Gamma_{tx}, \frac{\Gamma_x}{|\Gamma_x|}\rangle\, dx \\
&= \int_{S^1} \langle (\kappa N)_x, T\rangle\, dx \\
&= \int_{S^1} \langle \kappa_x N + \kappa N_x, T\rangle\, dx \\
&= \int_{S^1} \underbrace{\langle \kappa_x N, T\rangle}_{0 \text{ since } N\perp T} + \underbrace{\kappa\langle N_x, T\rangle}_{-\kappa^2 \text{ since } N_x = -\kappa T}\, dx \\
&= -\int_{S^1} \kappa^2\, dx
\end{aligned}$$

□

Propositions 2.1 and 2.1 establish that the Area and Length of Γ_t decrease with t. Thus the flow can only last for as long as both are strictly positive. In fact, according to Proposition 2.1, the area of the curve is strictly monotonically decreasing since $\frac{d}{dt}A(t) = -2\pi$ so $A(t) = -2\pi t + A(0)$. Resultantly, the flow can only last for $t \leqslant A(0)/2\pi$. Consequently, computing the variations of area and length provides us geometric information of the curves Γ_t; furthermore, the rather strong result for area provides an exact bound for how long curve shortening can last before the curve collapses to a point. Thus, as long as $A(0)$ is finite, the flow lasts for only a finite period of time. Another way to see this result is through the *avoidance principle.*

Theorem 2.3 (Avoidance Principle). *Let $X, Y : S^1 \times [0,T) \mapsto \mathbb{R}^2$ be closed embedded plane curves evolving by curve shortening such that $X(x,0) \neq Y(y,0)$ for any $x, y \in S^1$. Then $X(x,t) \neq Y(y,t)$ for any $(x,y,t) \in S^1 \times S^1 \times [0,T)$.*

Proof. The Avoidance Principle means that when two curves start disjoint, they stay disjoint if they evolve by the curve shortening flow. In fact, the proof of the Avoidance Principle reveals that not only do they stay disjoint but also that their distance *increases*. Recall that the distance between two sets $X, Y \subset \mathbb{R}^2$ is $\operatorname{dist}(X,Y) = \inf\{|x-y| \mid x \in X, y \in Y\}$. When X and Y are both compact, $\exists\,(x,y) \in X \times Y$ such that $\operatorname{dist}(X,Y) = |x-y|$. To that end, assume that $d_0 = \operatorname{dist}(X_0, Y_0) > 0$ and let $d : S^1 \times S^1 \times [0,T) \mapsto \mathbb{R}$ where $d(x,y,t) = |X(x,t) - Y(y,t)|$. We prove that for every $\varepsilon > 0$, $d(x,y,t)e^{\varepsilon(1+t)} > d_0$ for every $t \in [0,T)$. To that end, first observe that for $t = 0$ we have $d(x,y,0)e^{\varepsilon} \geqslant d_0 e^{\varepsilon} > d_0$. Now suppose towards a contradiction that $\exists\,(x,y,t) \in S^1 \times S^1 \times [0,T)$ such that $d(x,y,t)e^{\varepsilon(1+t)} \leqslant d_0$; then we have in particular that $D(t) := \min\{d(x,y,t)e^{\varepsilon(1+t)} \mid (x,y) \in S^1 \times S^1\} \leqslant d_0$ for some t; since $D(t)$ is continuous on $[0,T)$, we get from the IVT and the compactness of X_t and Y_t that $\exists\,(x_0,y_0,t_0)$ such that $D(t_0) = d(x_0,y_0,t_0)e^{\epsilon(1+t_0)} = d_0$. Moreover since $D(t) \leqslant d_0$ and is continuous over t we have the following analytic conditions. First, $D'(t_0) = (d(x,y,t)e^{\varepsilon(1+t)})_t \mid_{(x_0,y_0,t_0)} \leqslant 0$ since $D(t) > d_0$ on $[0,t_0)$. Second, since $D(t) = \min\{d(x,y,t)e^{\varepsilon(1+t)} \mid (x,y) \in S^1 \times S^1\} = e^{\varepsilon(1+t)}\min\{d(x,y,t) \mid (x,y) \in S^1 \times S^1\}$, we have that at (x_0,y_0), the first partial derivatives vanish, i.e $d_x(x_0,y_0,t_0) = d_y(x_0,y_0,t_0)$. Third, from the second-derivatives test, since $D(t_0)$ is a local minimum, the Hessian of D, $H(x,y,t) = \begin{vmatrix} \partial_x^2 d & \partial_{xy}^2 d \\ \partial_{xy}^2 d & \partial_y^2 d \end{vmatrix}$ is non-negative at (x_0,y_0,t_0). Now we compute each of these derivatives. We assume that X is parametrized by arc-length; note that this still leaves us free to choose a parametrization for Y.

$$\frac{\partial}{\partial x} d(x_0,y_0,t_0) = \frac{\partial}{\partial x}|X(x_0,t_0) - Y(y_0,t_0)| \tag{2.3}$$

$$= \frac{\langle X_x(x_0,t_0), X(x_0,t_0) - Y(y_0,t)\rangle}{|X(x_0,t_0) - Y(y_0,t_0)|} \tag{2.4}$$

$$= \langle X_x(x_0,t_0), \frac{X(x_0,t) - Y(y_0,t_0)}{|X(x_0,t_0) - Y(y_0,t_0)|}\rangle \tag{2.5}$$

$$= \langle T_X(x_0,t_0), w\rangle = 0 \tag{2.6}$$

$$\tag{2.7}$$

when $w = \frac{X(x_0,t_0)-Y(y_0,t_0)}{|X(x_0,t)-Y(y_0,t_0)|}$. A similar identity holds for $d_y(x_0,y_0,t_0)$ except that the sign is switched:

$$\frac{\partial}{\partial y} d(x_0,y_0,t_0) = -\langle T_Y(y_0,t_0), w\rangle = 0 \tag{2.8}$$

Consequently, both $T_Y(x_0,t_0)$ and $T_Y(y_0,t_0)$ are orthogonal to the same vector w. But now, if we parametrize Y in the right way (there are only two possible parametrizations here), $T_Y(x_0,t_0)$ and $T_Y(y_0,t_0)$ become parallel and since they are unit vectors we get $T_X = T_Y$. Next, we compute the second derivatives that arise in the Hessian via the chain rule. As a preliminary step, we differentiate $w(x,y,t) = \frac{X(x,t)-Y(y,t)}{|X(x,t)-Y(y,t)|}$ via the quotient rule

$$w_x = \frac{T_X|X(x,t) - Y(y,t)| - \langle T, w\rangle(X(x,t) - Y(y,t))}{|X(x,t) - Y(y,t)|^2} \tag{2.9}$$

$$= \frac{T_X - \langle T_X, w\rangle w}{|X - Y|} \tag{2.10}$$

Similarly,

$$w_y = -\frac{T_Y - \langle T_Y, w\rangle w}{|X - Y|} \tag{2.11}$$

We now use these in combination with the first derivatives to compute

$$\frac{\partial^2}{\partial x^2}d = \langle w_x, T_X\rangle + \langle w, (T_X)_x\rangle = \frac{T_X - \langle T_X, w\rangle w}{|X-Y|} + \langle w, \kappa_X N_X\rangle \tag{2.12}$$

$$\frac{\partial^2}{\partial y^2}d = \langle w_y, T_Y\rangle + \langle w, (T_Y)_y\rangle = \frac{T_Y - \langle T_Y, w\rangle w}{|X-Y|} - \langle w, \kappa_Y N_Y\rangle \tag{2.13}$$

$$\frac{\partial^2}{\partial xy}d = \langle w_y, T_X\rangle + \langle w_y, (T_X)_y\rangle = -\frac{\langle T_Y - \langle T_Y, w\rangle w, T_X\rangle}{|X-Y|} \tag{2.14}$$

Now, at (x_0, y_0, t_0) we have that $T_X = T_Y$; furthermore, they are perpendicular to w which for X is then the normal vector (since we chose the counterclockwise parametrization). It is also the normal vector for Y at (y_0, t_0). Consequently the above equations can be written in terms of curvature and d:

$$\frac{\partial^2}{\partial x^2}d = \frac{1}{d} + \kappa_X \quad \frac{\partial^2}{\partial y^2}d = \frac{1}{d} - \kappa_y \quad \frac{\partial^2}{\partial xy}d = -\frac{1}{d}$$

Since the Hessian is non-negative at (x_0, y_0, t_0) we have that

$$\frac{\partial^2}{\partial x^2}d + \frac{\partial^2}{\partial y^2}d - 2\frac{\partial^2}{\partial xy}d = \kappa_X - \kappa_Y \geqslant 0$$

Finally, it is time for the contradiction. The only derivative condition we haven't used is the one with respect to time. So let us calculate that. We have

$$0 \geqslant e^{-\varepsilon(1+t)}\frac{\partial}{\partial t}de^{\varepsilon(1+t)} \tag{2.15}$$

$$= \varepsilon d + e^{-\varepsilon(1+t)}e^{\varepsilon(1+t)}d_t \tag{2.16}$$

$$= \varepsilon d + \frac{\langle X_t - Y_t, X - Y\rangle}{|X-Y|} \tag{2.17}$$

$$= \varepsilon d + \langle X_t - Y_t, \frac{X-Y}{|X-Y|}\rangle \tag{2.18}$$

$$> \langle \kappa_X N_X - \kappa_Y N_Y, w\rangle \tag{2.19}$$

$$= \kappa_X - \kappa_Y \geqslant 0 \tag{2.20}$$

This is a contradiction where the last line follows because at (x_0, y_0, t_0), $w = N_X = N_Y$. At last, we have that $de^{\varepsilon(1+t)} > d_0$ for every $\varepsilon > 0$ implying that $d \geqslant d_0 > 0$. □

The proof of the avoidance principle gives us the following lesson: sometimes computing in $S^1 \times S^1 \times [0, T)$ is a good idea! Furthermore, the avoidance principle allows us to prove finite flow time in another way: let Γ be an embedded closed curve that undergoes curve shortening. Then since Γ_0 is bounded, we can contain it inside a large circle C_0^r of radius r centered at the origin. Due to the avoidance principle C_t^r and Γ_t never touch; but C_t^r shrinks to a point in finite time so the curve shortening flow also lasts on Γ for finite time. Note, however, that Γ_t may not necessarily shrink to a point too; there may be a point x and a sequence of times $\{t_n\}$ such that $\lim \kappa(x, t_n) = \infty$. In this case, x is termed a *blowup point* or a *singularity*. This is observed in the case of a curve with self-intersections, where loops collapse into cusps. One may ask how often curves blow up in this way; the answer for closed embedded curves is never! In 1986, Gage and Hamilton showed that all convex curves shrink to a round point; in 1987 Grayson proved that every non-convex embedded curve becomes convex, thus completing the celebrated **Gage-Hamilton-Grayson (GGH) Theorem** which says that closed embedded curves evolving by curve shortening always shrink to a point. In 1998, Huisken gave a short proof of the GGH theorem by identifying a functional that qunatified the "straightness" of a curve. More precisely, if Z is an embedded curve in $\mathbb{R}^2$ then its straightness $D(x, y)$ between two points $Z(x)$ and $Z(y)$ is the ratio of the distance between the points in the plane and their distance on the curve. In other words,

$$D(x, y) = \frac{|Z(y) - Z(x)|}{\int_x^y ds}$$

When Z evolves in time, we denote D as $D(x, y, y)$. Huisken showed that D at its global minimum over pairs of points on the curve is monotonic in time: this is the distance comparison principle. Huisken's main result involves showing that if the curve evolves into a singularity, then D becomes arbitrarily small violating its monotonicity.

3. Huisken's Distance Comparison Principles

Theorem 3.1. *Let $Z : I \times [0, T) \mapsto \mathbb{R}$ be a rectifiable non-closed curve evolving by curve shortening. Suppose D attains a local minimum at $(p, q) \in I \times I$ at time $t_0 \in [0, T)$. Then*

$$\frac{d}{dt} D(p, q, t_0) \geqslant 0$$

and equality holds if and only if Z is a straight line.

Proof. The proof of the distance comparison principle for non-closed curves follows almost the same strategy as the proof of the avoidance principle: at the local minimum we get an equality for first derivatives and inequality for second derivatives with respect to position. We compute these derivatives and using the inequalities find a bound for some quantity that shows up in the computation for the time derivative. To that end, let (p, q, t_0) be a local minimum over $I \times I$. Let $d = |Z(q, t_0) - Z(p, t_0)|$ and $l = \int_p^q ds$. Now let's compute the first and second derivatives! First, D_x:

$$D_x = \frac{\frac{\langle -Z_x, Z(y,t) - Z(x,t)\rangle}{|Z(y,t)-Z(x,t)|} \int_x^y ds - (-1)|Z(y) - Z(x)|}{(\int_x^y ds)^2} \tag{3.1}$$

$$= \frac{\langle -Z_x, \frac{Z(y,t)-Z(x,t)}{|Z(y,t)-Z(x,t)|}\rangle \int_x^y ds + |Z(y, t) - Z(x, t)|}{(\int_x^y ds)^2} \tag{3.2}$$

$$= \frac{\langle -Z_x, w\rangle \int_x^y ds + |Z(y, t) - Z(x, t)|}{(\int_x^y ds)^2} \tag{3.3}$$

$$= \frac{\langle -Z_x, w\rangle}{\int_x^y ds} + \frac{|Z(y, t) - Z(x, t)|}{(\int_x^y ds)^2} \tag{3.4}$$

Here $w = \frac{Z(y,t)-Z(x,t)}{|Z(y,t)-Z(x,t)|}$. Note that at (p, q, t_0) we have that $D_x(p, q, t_0) = \frac{\langle -Z_x(p,t_0), w\rangle}{l} + \frac{d}{l^2} = 0$ so $\langle Z_x(p, t_0), w\rangle = d/l$. The computation for D_y is almost identical except the signs are flipped since $\frac{d}{dy}\int_x^y ds = 1$ instead of -1 and $|Z(y, t) - Z(x, t)|_y = \langle Z_y, w\rangle$ (for the x variable, $|Z(y, t) - Z(x, t)|_x = \langle -Z_x, w\rangle$).

$$D_y = \frac{\langle Z_y, w\rangle}{\int_x^y ds} - \frac{|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^2} \tag{3.5}$$

Consequently, at (p, q, t_0) we again get that $D_y(p, q, t_0) = d/l$. To compute the second derivatives, we first make a preliminary computation of w_x and w_y:

$$w_x = \frac{-Z_x|Z(y,t) - Z(x,t)| - \langle -Z_x, w\rangle(Z(y,t) - Z(x,t))}{|Z(y,t) - Z(x,t)|^2} \tag{3.6}$$

$$= \frac{-Z_x + \langle Z_x, w\rangle \frac{Z(y,t)-Z(x,t)}{|Z(y,t)-Z(x,t)|}}{|Z(y,t) - Z(x,t)|} \tag{3.7}$$

$$= \frac{-Z_x + \langle Z_x, w\rangle w}{|Z(y,t) - Z(x,t)|} \tag{3.8}$$

Similarly,

$$w_y = \frac{Z_y - \langle Z_y, w\rangle}{|Z(y,t) - Z(x,t)|} \tag{3.9}$$

Now onto computing the second derivatives:

$$D_{xx} = \frac{(\langle -Z_{xx}, w_x\rangle + \langle -Z_x, w\rangle)\int_x^y ds - (-1)\langle -Z_x, w\rangle}{(\int_x^y ds)^2} \tag{3.10}$$

$$+ \frac{\langle -Z_x, w\rangle(\int_x^y ds)^2 - 2\int_x^y ds(-1)|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^4} \tag{3.11}$$

$$= \frac{\langle -Z_{xx}, w\rangle}{(\int_x^y ds)} - \frac{\langle Z_x, w_x\rangle}{\int_x^y ds} - \frac{\langle Z_x, w\rangle}{(\int_x^y ds)^2} - \frac{\langle Z_x, w\rangle}{(\int_x^y ds)^2} + 2\frac{|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^3} \tag{3.12}$$

$$= -\frac{\langle Z_{xx}, w\rangle}{(\int_x^y ds)} - \frac{\langle Z_x, \frac{-Z_x + \langle Z_x, w\rangle w}{|Z(y,t)-Z(x,t)|}\rangle}{\int_x^y ds} - 2\frac{\langle Z_x, w\rangle}{(\int_x^y ds)^2} + 2\frac{|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^3} \tag{3.13}$$

$$= -\frac{\langle Z_{xx}, w\rangle}{(\int_x^y ds)} - \frac{\langle Z_x, -Z_x\rangle + \langle Z_x, w\rangle^2}{|Z(y,t) - Z(x,t)|\int_x^y ds} - 2\frac{\langle Z_x, w\rangle}{(\int_x^y ds)^2} + 2\frac{|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^3} \tag{3.14}$$

$$= -\frac{\langle Z_{xx}, w\rangle}{(\int_x^y ds)} + \frac{1 - \langle Z_x, w\rangle^2}{|Z(y,t) - Z(x,t)|\int_x^y ds} - 2\frac{\langle Z_x, w\rangle}{(\int_x^y ds)^2} + 2\frac{|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^3} \tag{3.15}$$

At (p, q, t_0), $\langle Z_x, w\rangle = d/l$ so

$$D_{xx}(p, q, t_0) = -\frac{\langle Z_{xx}, w\rangle}{l} + \frac{1 - \frac{d^2}{l^2}}{dl} - 2\frac{d/l}{l^2} + 2\frac{d}{l^3} \tag{3.16}$$

$$= -\frac{\langle Z_{xx}, w\rangle}{l} + \frac{1}{dl} - \frac{d}{l^3} \tag{3.17}$$

The computation for D_{yy} is almost exactly the same except that the sign flippage is almost divinely convenient:

$$D_{yy} = \frac{(\langle Z_{yy}, w\rangle + \langle Z_y, w_y\rangle)\int_x^y ds - \langle Z_y, w\rangle}{(\int_x^y ds)^2} \tag{3.18}$$

$$- \frac{\langle Z_y, w\rangle(\int_x^y ds)^2 - 2\int_x^y ds|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^4} \tag{3.19}$$

$$= \frac{\langle Z_{yy}, w\rangle}{(\int_x^y ds)} + \frac{\langle Z_y, w_y\rangle}{\int_x^y ds} - \frac{\langle Z_y, w\rangle}{(\int_x^y ds)^2} - \frac{\langle Z_y, w\rangle}{(\int_x^y ds)^2} + 2\frac{|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^3} \tag{3.20}$$

$$= \frac{\langle Z_{yy}, w\rangle}{(\int_x^y ds)} + \frac{\langle Z_y, \frac{Z_y - \langle Z_y, w\rangle w}{|Z(y,t)-Z(x,t)|}\rangle}{\int_x^y ds} - 2\frac{\langle Z_y, w\rangle}{(\int_x^y ds)^2} + 2\frac{|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^3} \tag{3.21}$$

$$= \frac{\langle Z_{yy}, w\rangle}{(\int_x^y ds)} + \frac{\langle Z_y, Z_y\rangle + \langle Z_y, w\rangle^2}{|Z(y,t) - Z(x,t)|\int_x^y ds} - 2\frac{\langle Z_y, w\rangle}{(\int_x^y ds)^2} + 2\frac{|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^3} \tag{3.22}$$

$$= \frac{\langle Z_{yy}, w\rangle}{(\int_x^y ds)} + \frac{1 - \langle Z_y, w\rangle^2}{|Z(y,t) - Z(x,t)|\int_x^y ds} - 2\frac{\langle Z_y, w\rangle}{(\int_x^y ds)^2} + 2\frac{|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^3} \tag{3.23}$$

At (p, q, t_0), $\langle Z_x, w\rangle = d/l$ so

$$D_{xx}(p, q, t_0) = \frac{\langle Z_{yy}, w\rangle}{l} + \frac{1 - \frac{d^2}{l^2}}{dl} - 2\frac{d/l}{l^2} + 2\frac{d}{l^3} \tag{3.24}$$

$$= \frac{\langle Z_{yy}, w\rangle}{l} + \frac{1}{dl} - \frac{d}{l^3} \tag{3.25}$$

Thus, we see that between D_{xx} and D_{yy} only *one* sign flips: the one on the all important curvature term $\langle Z_{yy}, w\rangle/l/$ Finally, we compute $D_{xy} = D_{yx}$:

$$D_{yx} = \frac{(\langle Z_{xy}, w\rangle + \langle Z_y, w_x\rangle)\int_x^y ds - (-1)\langle Z_y, w\rangle}{(\int_x^y ds)^2} \tag{3.26}$$

$$- \frac{\langle -Z_x, w\rangle(\int_x^y ds)^2 - 2\int_x^y ds(-1)|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^4} \tag{3.27}$$

$$= \frac{\langle Z_y, w_x\rangle}{\int_x^y ds} + \frac{\langle Z_x, w\rangle}{(\int_x^y ds)^2} + \frac{\langle Z_y, w\rangle}{(\int_x^y ds)^2} + 2\frac{|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^3} \tag{3.28}$$

$$= \frac{\langle Z_y, \frac{-Z_x + \langle Z_x, w\rangle w}{|Z(y,t)-Z(x,t)|}\rangle}{\int_x^y ds} + \frac{\langle Z_x, w\rangle}{(\int_x^y ds)^2} + \frac{\langle Z_y, w\rangle}{(\int_x^y ds)^2} + 2\frac{|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^3} \tag{3.29}$$

$$= \frac{\langle Z_y, -Z_x\rangle}{|Z(y,t) - Z(x,t)|\int_x^y ds} + \frac{\langle Z_x, w\rangle\langle Z_y, w\rangle}{|Z(y,t) - Z(x,t)|\int_x^y ds} \tag{3.30}$$

$$+ \frac{\langle Z_x, w\rangle}{(\int_x^y ds)^2} + \frac{\langle Z_y, w\rangle}{(\int_x^y ds)^2} + 2\frac{|Z(y,t) - Z(x,t)|}{(\int_x^y ds)^3} \tag{3.31}$$

Computing at (p, q, t_0) we have:

$$D_{yx}(p,q,t_0) = -\frac{\langle Z_y, Z_x\rangle}{dl} + \frac{d^2/l^2}{dl} + \frac{d/l}{l^2} + \frac{d/l}{l^2} + 2\frac{d}{l^3} \tag{3.32}$$

$$= -\frac{\langle Z_y, Z_x\rangle}{d} + \frac{5d}{l^3} \tag{3.33}$$

Finally, computing the Hessian $H(p,q,t_0) = D_{xx} + D_{yy} - 2D_{xy} \geqslant 0$ we get

$$H(p,q,t_0) = -\frac{\langle Z_{xx}, w\rangle}{l} + \frac{1}{dl} - \frac{d}{l^3} + \frac{\langle Z_{yy}, w\rangle}{l} + \frac{1}{dl} - \frac{d}{l^3} + 2\frac{\langle Z_y, Z_x\rangle}{d} - \frac{10d}{l^3} \tag{3.34}$$

$$= \frac{1}{l}\langle Z_{yy} - Z_{xx}, w\rangle - \frac{10d}{l^3} + \frac{1}{d}\Big(\frac{2}{l} + \frac{1}{2}|Z_y + Z_x| - \frac{1}{2}|Z_y - Z_x|\Big) \geqslant 0 \tag{3.35}$$

But the last two terms on the right are both negative (one by choosing an appropriate parametrization and the other by observing that $l > d$ at a local minimum and $Z_x + Z_y$ is parallel to w). Consequently,

$$\frac{1}{l}\langle Z_{yy} - Z_{xx}, w\rangle \geqslant 0 \tag{3.36}$$

At last, we compute the time derivative of D:

$$\frac{d}{dt}D(p,q,t) = \frac{d}{dt}\frac{|Z(x,t) - Z(y,t)|}{\int_x^y ds} \tag{3.37}$$

$$= \frac{\langle Z_t(x,t) - Z_t(y,t), w\rangle \int_x^y ds - |Z(x,t) - Z(y,t)|\frac{d}{dt}\int_x^y ds}{(\int_x^y ds)^2} \tag{3.38}$$

$$= \frac{\langle Z_{yy} - Z_{xx}, w\rangle}{\int_p^q ds} - \frac{d}{l^2}\frac{d}{dt}\int_p^q ds \tag{3.39}$$

$$\geqslant -\frac{d}{l^2}\frac{d}{dt}\int_p^q ds \tag{3.40}$$

Here the last step follows from Equation 3.36. Finally, from Proposition 2.2 we have that $(d/dt)ds = -\kappa^2\, du$; as a consequence we have that $-\frac{d}{l^2}\frac{d}{dt}\int_p^q ds = \frac{d}{l^2}\int_x^y -\kappa^2\, du \geqslant 0$. Combining this result with Equation 3.40 we have that $\frac{d}{dt}D(p,q,t) \geqslant 0$. Furthermore, equality holds if and only if all the inequalities are equalities; this would require $\int_p^q \kappa^2\, du$ to be 0 and hence for $\kappa = 0$ for $u \in [p,q]$. Consequently, Z had to be a straight line. □

Huisken's distance comparison principle cannot be applied straightaway to closed curves because then D would become periodic over $I \times I$; furthermore $l(x,y,t)$ is only smoothly defined up to halfway around the length of the curve (because there are always two possible lengths between two points on a closed curve and l represents the shortest one). As a consequence, the quantity l is replaced by $\psi = \frac{L}{\pi}\sin(\frac{l\pi}{L})$ where L is the length of Z. The distance comparison principle for closed curves then applies to the functional $F = |Z(x,t) - Z(y,t)|/\psi$.

4. Application to curves with 1 immersed disk

In this section we explore a way to use Huisken's measure of straightness to study the problem of a curve with an immersed disk. Note that $X : \overline{D}^2 \mapsto \mathbb{R}^2$ is an immersion whenever $|J_X| \neq 0$ where J_0 is the Jacobian of X. Then $X(\partial\overline{D}^2)$ is a curve in $\mathbb{R}^2$ and is said to bound an immersed disk. One such example of a curve is the following class of curves we term "crab curves"

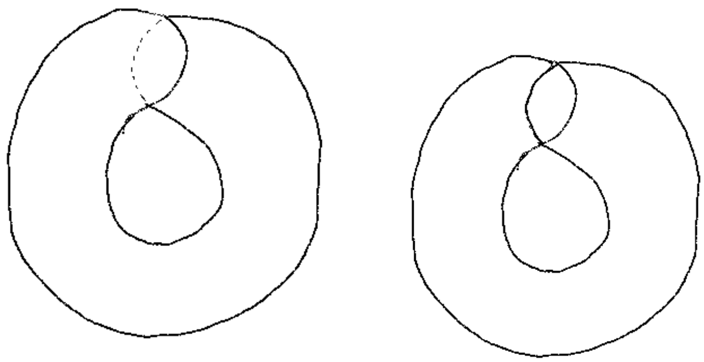

FIGURE 2. A disk immersed in the plane (left) and the corresponding curve (right)

Here the immersion is given by overlapping one "arm" of the disk over the other (like a crab) and the curve is obtained by tracking the boundary of the disk all the way under the overlap. We ask the following question: if we evolve the curve through curve shortening, when does the flow preserve the immersed disk? The answer seems to depend on the size of the overlap with respect to the size of the cavity (or the ratio of the upper bulb to the lower bulb of the figure-eight seen in the curve). On one hand, when the overlap is significantly larger than the cavity, a singularity develops:

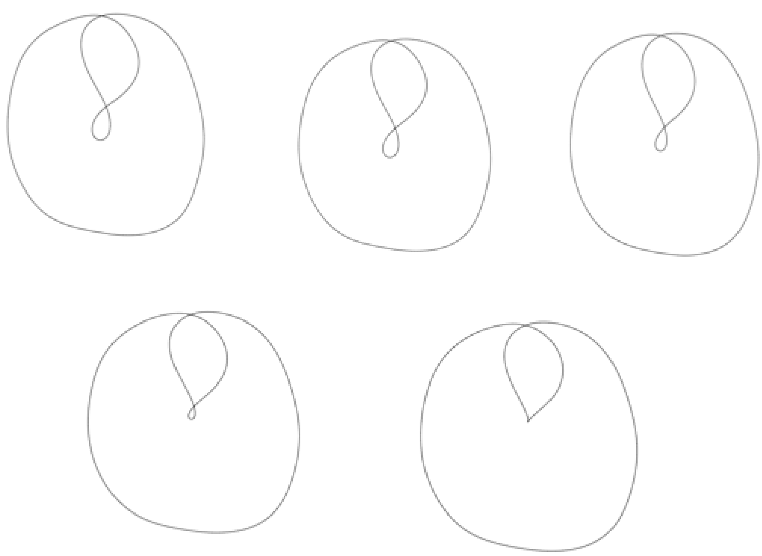

FIGURE 3. A crab curve which develops a singularity

On the other hand, when the cavity is larger than the overlap, then the curve evolves smoothly:

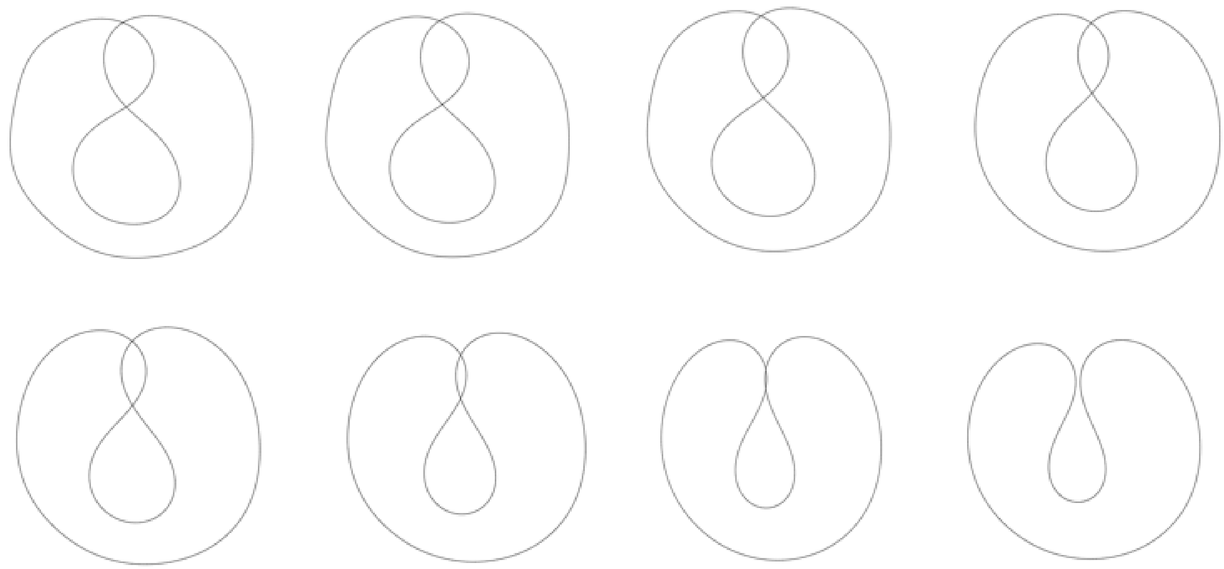

FIGURE 4. A crab curve which evolves smoothly

To possibly classify the crab curves which shorten smoothly, we suggest the following three strategies:

(1) **The area ratio of the figure-eight:** Let A and B denote the upper and lower bulbs of the figure-eight found in the crab curve. Then the *area ratio* $R = area(A)/area(B)$. Experimentally, it seems that when $R < 1$ the curve develops a singularity and when $R > 1$ it evolves smoothly. In the singular cases $R(t) \to S$ for some constant $S > 0$ as $t \to T$, the blowup time; on the other hand in the smooth cases $R(t)$ approaches 0 as the two self-intersections of the curve resolve without it developing a singularity.

(2) **Computing** $F(x, y, t)$**:** Instead of studying a global property like the area ratio, we may also want to consider local properties like the straightness $F(x, y, t)$ for pairs of points on the curve. In fact, intuitively, $F(x, y, t)$ should reach a local maximum on the points of B that determine its diameter; the only issue is that the segment connecting is entirely outside the disk. A more refined approach would be to restrict F to $(x, y) \in \Gamma \times \Gamma$ such that if s is the segment joining $\Gamma(x)$ and $\Gamma(y)$ then $X^{-1}(s)$ is a chord in $\overline{D}^2$.

(3) **Hairclips**: Our experiments reveal that as the curve evolves, the two arms detract from each other and the cavity becomes less concave (due to curve shortening). Consequently, the curve has two competing aspects: the shortening cavity and the withdrawing arms. In fact, the withdrawing arms seem to act as solitons. Consequently, it could be fruitful to address examples of curves which have local regions that behave like solitons and compare them to crab curves.

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