NOTES ON NOETHERIAN RINGS

Definition 1. A ring R is said to be *Noetherian* if there are no infinite ascending chains of ideals $I_i \subset R$ of the form

$$I_1 \subset I_2 \subset \dots$$

Equivalently, all ascending chains of the form

$$I_1 \subseteq I_2 \subseteq \dots$$

terminate, i.e there is an integer k such that

$$I_1 \subseteq I_2 \subseteq \dots I_k = I_{k+1} = I_{k+2} \dots$$

Theorem 1. *A ring R is Noetherian iff every ideal in R is finitely generated, i.e $\forall I \subset R$ such that I is an ideal, $\exists r_1, \dots, r_n \in R$ such that $I = \langle r_1 \dots r_n \rangle$*

Proof. ($\Rightarrow$) Assume R is Noetherian, and that there is an ideal that isn't finitely generated. Pick $r_1 \in I$. Since $I - r_1 \neq \varnothing$ (if it wasn't we would have that I was finitely generated), $\langle r_1 \rangle \neq I$. Now, we will use a similar construction to generate an infinite ascending chain. Given $r_1, \dots r_s \in I$, pick $r_{s+1} \in I - \langle r_1, \dots r_s \rangle$. Then,

$$\langle r_1 \rangle \subset \langle r_1, r_2 \rangle \subset \langle r_1, \dots, r_s \rangle \subset \dots$$

Hence we have generated an infinite ascending chain of ideals in R. But this is a contradiction, since R is Noetherian. Thus, if R is Noetherian, every ideal is finitely generated.
($\Leftarrow$) Now suppose every ideal $I \subset R$ is finitely generated. Suppose

$$I_1 \subseteq I_2 \subseteq \dots$$

is an infinite ascending chain of ideals. Let $I = \bigcup_{i=1}^{\infty} I_i$. Then I is an ideal since $0 \in I_i$ and $\forall a, b \in I$, $a \in I_k$ and $b \in I_j$ for some $i, j \in \mathbb{N}$. Since I_k and I_j are part of an ascending chain, without loss of generality assume that $I_k \subset I_j$. Then $a \in I_j$ and $a + b \in I_j \subset I$. Finally, $\forall r \in R$ and $i \in I$, $i \in I_k$ for some $k \in \mathbb{N}$. Since I_k is an ideal, $ir = ri \in I_k \subset I$. Since I is an ideal in R, by our assumption, I is finitely generated. Let $I = \langle r_1, \dots, r_s$. Then $\forall r_j, \exists I_{i_j}$ s.t $r_j \in I_{i_j}$. Now pick $k = \max i_1, \dots, i_s$. Here k is the highest index among the indices of the generator-containing ideals I_{i_j}. We can pick it because the set of indices is finite. Thus, $r_1, \dots, r_s \in I_k$ and $I = \langle r_1, \dots, r_s \rangle \subseteq I_k \subseteq I_{k+1} \subseteq \dots$. Also, $I_{k+l} \subseteq I$, $\forall\, l \in \mathbb{N}$. As a result,

$$I = I_{k+1} = I_{k+l}$$

Thus, the infinite ascending chain

$$I_1 \subseteq I_2 \subseteq \dots \subseteq I_k = I_{k+1} = I_{k+l}$$

terminates. $\square$

Theorem 2. *Let R be a Noetherian ring and I an ideal of R. Then*

(1) *R/I is Noetherian*
(2) *R^n is Noetherian for $n \in \mathbb{N}$*

Proof. (1) From the correspondence theorem, we know that there is an inclusion preserving, one-to-one correspondence between the ideals of R/I and the ideals of R containing I. Thus, consider any ascending chain of ideals of R/I,

$$\overline{I_1} \subseteq \overline{I_2} \subseteq \dots$$

Then there is a one-to-one between this chain of ideals and the chain of ideals of R given by

$$I_1 \subseteq I_2 \subseteq \dots$$

Since R is Noetherian, this chain stabilizes. Since there is an inclusion-preserving correspondence between this chain and the proposed chain of ideals of R/I, the ascending chain of ideals of R/I also stabilizes.
(2) We will prove this by induction. The base case is painfully trivial. $R^1 \cong R$ and R is Noetherian so R^1 is Noetherian. For the inductive hypothesis, assume that $\forall n : 1 \leq n \leq k$, R^n is Noetherian. For the inductive step, we need to prove that R^{k+1} is Noetherian. To that end, consider an ascending chain of ideals of R^{k+1}:

$$I_1 \subseteq I_2 \subseteq \dots \subseteq I_s \subseteq \dots$$

Consider any I_s in this ascending chain and separate it into vectors of k components and the $k+1$th component. Let $I_{s,1} = \{(x_1, \cdots, x_k) \,|\, (x_1, \cdots, x_k, x_{k+1}) \in I_s \text{ for some } x_{k+1} \in R\}$ and $I_{s,2} = \{x_i \,|\, (x_1, \cdots, x_k, x_i) \in I_s \text{ for some } (x_1, \cdots, x_k) \in R^k\}$. Now we show that the $I_{s,1}$'s and the $I_{s,2}$'s form ascending chains. First of all, $I_{s,1}$ and $I_{s,2}$ are ideals for all s. The proof is as follows. Since I_s is an ideal, $(0, \dots, 0) \in I_s$. As a result, $(0, \dots, 0) \in I_{s,1}$ and $0 \in I_{s,1}$. Now, suppose $(a_1, \dots, a_k), (b_1, \dots, b_k) \in I_{s,1}$. Then $\exists a, b \in I_s$ s.t $a = (a_1, \dots, a_k, a_{k+1})$ and $b = (b_1, \dots, b_k, b_{k+1})$. Since I_s is an ideal, $a + b = (a_1 + b_1, \dots a_k + b_k, a_{k+1} + b_{k+1}) \in I_s$. Taking the first k components, we conclude that $(a_1 + b_1, \dots a_k + b_k) \in I_{s,1}$. Finally, $\forall r \in R^k and$ □

Theorem 3. ***Hilbert Basis Theorem***
Let R be a ring and let $R[x]$ be the polynomial ring of R. If R is Noetherian then $R[x]$ is Noetherian.

Proof. We will prove that all ideals of $R[x]$ are finitely generated. From Theorem 1, this is equivalent to $R[x]$ being Noetherian. Let $I \subset R[x]$ be an ideal. Define $I_i = \{f \in I : \deg(f) = i\}$. Then $I_i' = \{\text{lc}(f) : f \in I_i\} \cup 0$ is an ideal of R, because

(1) $\forall a, b \in I_i'$, $\exists f, g \in I_i$ s.t $f(x) = ax^i + f'(x)$ and $g(x) = bx^i + g'(x)$. $f(x) + g(x) = (a+b)x^i + f'(x) + g'(x) \in I_i$. As a result, $a + b \in I_i'$.
(2) $0 \in I_i'$ by definition
(3) Let $r \in R$ and $j \in I_i'$. Then $\exists\, f(x) \in I_i$ s.t $f(x) = jx^i + f'(x)$. Thus, $rf(x) = (rj)x^i + rf'(x) \in I_i$. Consequently, $rj = jr \in I_i'$

Note that $I_i' \subset I_{i+1}'$ because if $a \in I_i'$, $\exists\, f(x) = ax^i + f'(x)$. Since I is an ideal, $xf(x) = ax^{i+1} + xf'(x) \in I$. $\deg(xf(x)) = i + 1$ so $a \in I_{i+1}'$. We have thus produced a chain of ideals of R such that

$$I_0' \subseteq I_1' \subseteq I_2' \subseteq \dots$$

Since R is Noetherian, this chain terminates. Thus $\exists\, I'_l$ such that $I'_l = I'_{l+s}$. Since the I'_i's are all finitely generated, let $I'_i = \langle a_{i,1}, \ldots, a_{i,n_i} \rangle$. Let $f_{i,j} \in I_i$ s.t $f_{i,j} = a_{i,j}x^i + f'_{i,j}(x)$. We prove using induction that I is generated by $\{f_{i,j} : 0 \leq i \leq l,\, 0 \leq j \leq n_i\}$.
Base Case: $f = c \in I_0 = I'_0$ so c can be expressed as a combination of $a_{0,1}, \ldots, a_{0,n_0}$ since every ideal in R is finitely generated. For the inductive hypothesis, assume that all polynomials with degree less than k are generated by the proposed set, i.e for $\deg(f) < k$, $f \in \langle \{f_{i,j} : 0 \leq i \leq l,\, 0 \leq j \leq n_i\} \rangle$. For the inductive step, let $f \in I$ such that $\deg(f) = k$.
Case 1: $k \leq l$**.** $f \in I_k$ and $\mathrm{lc}(f) = a \in I'_k$ so $a = c_1 a_{k,1} + \ldots + c_{n_k} a_{k,n_k}$ for some $c_1, \ldots, c_{n_k} \in R$. Set $f' = f - c_1 f_{k,l} - \ldots - c_{n_k} f_{k,n_k}$. Now, the linear combinations of all the $f_{k,i}$'s with c_i's produce a k-degree polynomial whose leading coefficient is the same as that of f. Thus, by subtracting the linear combination from f, we produce a polynomial f' that is reduced by one degree. Thus, $\deg(f') < k$ and by the inductive hypothesis, we have that $f' \in \langle \{f_{i,j} : 0 \leq i \leq l,\, 0 \leq j \leq n_i\} \rangle$. Since, $f = f' + c_1 f_{k,l} + \ldots + c_{n_k} f_{k,n_k}$, we have that $f \in \langle \{f_{i,j} : 0 \leq i \leq l,\, 0 \leq j \leq n_i\} \rangle$.
Case 2: $l < k$**.** Let $f \in I_k$. Then $\mathrm{lc}(f) = a \in I'_k = I'_l$. Now, exactly as in Case 1, we express a in terms of $a_{l,1}$ through a_{l,n_l}. Thus, $a = c_1 a_{l,1} + \ldots + c_{n_l} a_{l,n_l}$. Let $f' = f - c_1 f_{l,1} - \ldots = c_{n_l} f_{l,n_l}$. Then $\deg(f') < k$, which means $f' \in \langle \{f_{i,j} : 0 \leq i \leq l,\, 0 \leq j \leq n_i\} \rangle$, which in turn means that $f \in \langle \{f_{i,j} : 0 \leq i \leq l,\, 0 \leq j \leq n_i\} \rangle$.
In conclusion, every ideal of $R[x]$ was shown to be finitely generated. As a result, $R[x]$ is Noetherian. □

Definition 2. Let $(R, +, *)$ be a commutative ring with unity. An *R-Module M* over the ring R is defined as a non-empty set such that $(M, +)$ is an Abelian group and $\exists$ an function $\cdot : R \times M \mapsto M$ such that $\forall\, x, y \in M$ and $r, s \in R$,

(1) $r \cdot (s \cdot x) = (rs) \cdot x$
(2) $r \cdot (x + y) = r \cdot x + r \cdot y$
(3) $(r + s) \cdot x = r \cdot x + s \cdot x$
(4) $1 \cdot x = x$

Note that Property (4) is equivalent to $0 \cdot x = 0$, so we could have had that as Property (4) instead.