What do multiple Walrasian equilibria mean?

Shashank Sule

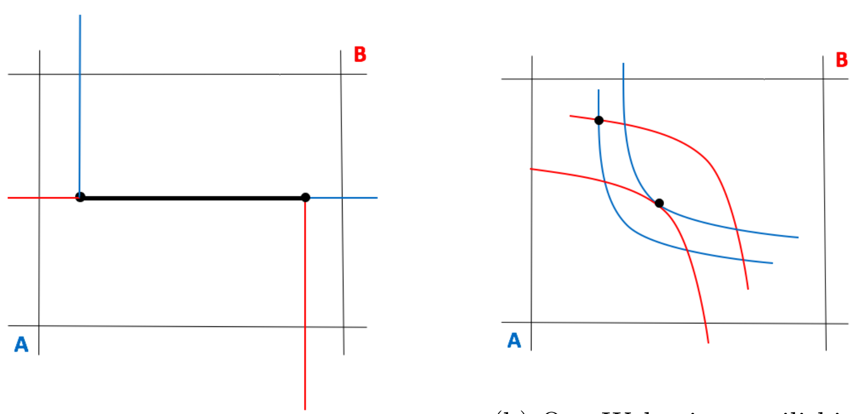
November 2017, ECON 301

Abstract

I use an example to demonstrate that finitely many Walrasian equilibria can arise in a 2 agent 2 good general equilibrium model if one agent has rising demand with respect to price. Furthermore, I argue that multiple Walrasian equilibria present an allocation challenge that cannot be met with the standard assumptions of general equilibrium theory. Finally, I show an application of multiple equilibria: The Transfer paradox.

Introduction

A Walrasian equilibrium can be understood as the outcome of a pure exchange economy where both agents are left weakly better off from their initial endowments and where the total demand for each good equals its total supply. For a given set of endowments, a Walrasian equilibrium can be unique, such as in the case of Cobb-Douglas preferences. Assuming perfectly divisible goods, it can also be a continuous set, such as in the case of Leontief (perfect compliments) preferences.



(a) Many Walrasian equilibria

(b) One Walrasian equilirbium

Figure 1: Walrasian equilibria in Leontief (left) and Cobb-Douglas (right) preferences

But there is also a case where there are finitely many Walrasian equilibria. I demonstrate this through a historical example that takes us back to 1929, 11 years after the end of the first World War.

Multiple Walrasian equilibria

I present the case of multiple Walrasian equilibria by considering an exchange economy between Germany and England. Germany and England both have some endowments of Pound Sterling and Reichmark, (s, m). Germany consumes in Reichmark while England consumes in Pound Sterling. They have quasilinear, strictly convex and strongly monotonic utility functions, where both agents have weakly positive endowments of both Pound and Reichmark[1]:

$$u_G(s, m) = s - \frac{1}{4}(m)^{-4}$$

$$u_E(s, m) = -\frac{1}{4}(s)^{-4} + m$$

Without a loss of generality, I normalize p_m to 1, so that I can eventually express the Marshallian demands as a function of $p_s only$. Finally, I (somewhat contrivedly) set Germany's endowment to $(2, r)$ and England's endowment to $(r, 2)$, where $r = 2^{4/5} - 2^{1/5}$.

I compute Germany's Marshallian demand for Reichmark by equating the ratio of the Marginal Rates of Substitution to the price ratio.

$$\frac{1}{(m)^{-5}} = \frac{p_s}{1}$$

$$\implies m = (p_s)^{1/5}$$

Now, I find Marshallian demand for Pound by plugging in $m = (p_s)^{1/5}$ into Germany's budget constraint

$$p_s s^G + p_m m^G = p_s s + p_m m$$

Rearranging,

$$p_s s = p_s s^G + p_m m^G - p_m m$$

[1] I modified an example in MWG to obtain these utility functions

Dividing the equation by p_s

$$s = \frac{p_s}{p_s}s^G + \frac{p_m}{p_s}m^G - \frac{p_m}{p_s}m$$

$$\implies s = s^G + \frac{p_m}{p_s}m^G - (\frac{p_m}{p_s})(\frac{p_s}{p_m})^{1/5}$$

$$\implies s = 2 + \frac{1}{p_s}r - (\frac{1}{p_s})^{4/5}$$

Non-negativity of Marshallian demand is violated if

$$2 + \frac{1}{p_s}r < (\frac{1}{p_s})^{4/5}$$

In that case, there is a corner solution. Then, either $s = 0$ or $m = 0$. But if $m = 0$, $u_G(s, m) = -\infty$. The corner solution is where

$$s = 0 \text{ and } m = \frac{p_s s^G + p_m m^G}{p_m} = 2p_s + r$$

Thus, the Marshallian demands for Germany are [2]

$$d_s^G(p_s) = \begin{cases} 2 + \frac{1}{p_s}r - (\frac{1}{p_s})^{4/5} & 2 + \frac{1}{p_s}r \geq (\frac{1}{p_s})^{4/5} \\ 0 & \text{otherwise} \end{cases}$$

$$d_m^G(p_s) = \begin{cases} p_s^{1/5} & 2 + \frac{1}{p_s}r \geq (\frac{1}{p_s})^{4/5} \\ 2p_s + r & \text{otherwise} \end{cases}$$

Since England has a symmetric utility function, the exact same Marshallian demand analysis applies. Here are the Marshallian demands for Germany and England assuming interior solutions:

To find the Walrasian Equilibrium, I set total demand for s equal to total supply, or the total endowment which is $2 + r$

[2] The general form of the Marshallian demands is:

$$d_s^G(p_s, p_m, p_s s^G + p_m m^G) = \begin{cases} s^G + \frac{p_m}{p_s}m^G - (\frac{p_s}{p_m})^{-4/5} & s^G + (\frac{p_m}{p_s})m^G \geq (\frac{p_s}{p_m})^{-4/5} \\ 0 & \text{otherwise} \end{cases}$$

$$d_m^G(p_s) = \begin{cases} (p_s/p_m)^{1/5} & s^G + (\frac{p_m}{p_s})m^G \geq (\frac{p_s}{p_m})^{-4/5} \\ \frac{p_s s^G + p_m m^G}{p_m} & \text{otherwise} \end{cases}$$

Parameter	Germany	England
Endowment	$(2, r)$	$(2, r)$
$d_s(p_s)$	$2 + \frac{1}{p_s}r - (\frac{1}{p_s})^{4/5}$	$p_s^{-1/5}$
$d_m(p_s)$	$p_s^{1/5}$	$2 + rp_s - (p_s)^{4/5}$

Table 1: Marshallian Demands of Germany and England as a function of p_s

$$2 + \frac{1}{p_s}r - (\frac{1}{p_s})^{4/5} + p_s^{-1/5} = 2 + r$$

For simplicity, I relabel p_s as p. Rearranging, and factoring out r we get

$$r(\frac{1}{p} - 1) - (\frac{1}{p})^{4/5} + (\frac{1}{p})^{1/5} = 0$$

Note that the expression on the left is the Excess Demand function, E. It is defined as total demand minus the total supply and is a function of price, as the total demand is a function of price and supply is fixed. When $E = 0$, total demand equals total supply. Thus, p such that $E(p) = 0$ is the Walrasian equilibrium price. To find the Walrasian equilibrium prices, I find the intercepts of the above equation.

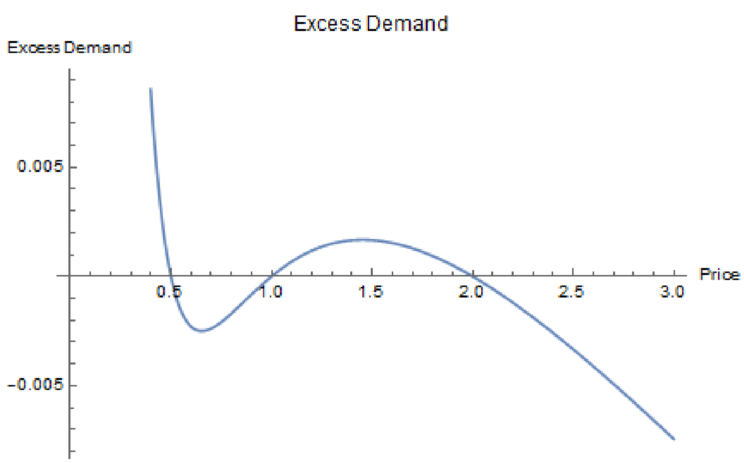


Figure 2: Excess Demand

Curiously, the excess demand curve has three solutions: 0.5, 1 and 2! Thus, there are three Walrasian equilibria.[3]. Furthermore, demand is locally *upward sloping* when $p \in [0.65, 1.45]$. To explain why, here are the two Marshallian demands for Germany and England:

[3]These prices check out the interior solution constraint, and the Walrasian equilibrium constraint. Also, this is where the peculiar value of r came in handy. It gave us nice solutions

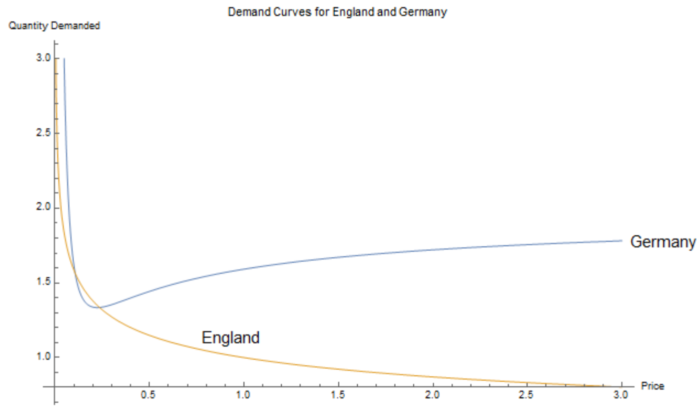


Figure 3: Marshallian Demands

For sufficiently high prices of Pound, German demand *increases*. Why could this be? To answer this question, it helps to first look at the England's demand for the Pound. England consumes in the pound, so it faces diminishing returns in the pound. On the other hand, Germany faces diminishing returns in the Reichmark. When the Pound's price changes at sufficiently high levels, Germany substitutes to consuming more Reichmark. But the substitution is negligible, due to diminishing returns. After consuming Reichmark, Germany spends the remainder of its wealth on Pound. However, its wealth increases quite appreciably in comparison. Consequently, its consumption of Pound also increases. Essentially, the income effect dominates the substitution effect. Germany substitutes to having more Reichmark but the income effect is so strong that it consumes more pound too.

But the Pound is not a giffen good. Giffen goods are *inferior* goods for whom the income effect dominates the substitution effect. For inferior goods, the individual's decrease in effective wealth is termed the income effect. But in the case of the pound, the income effect implies an increase in Germany's wealth.

In fact, for any case of multiple Walrasian equilibria, it must be that at least one agent has an upward sloping demand .

Conjecture 1. *If a multi agent, two good exchange economy in general equilibrium exhibits distinct and finite Walrasian Equilibria, there is at least one agent i with demand $d_i(p_k)$ for good k where $d_i'(p_k) > 0$ for some $p_k > 0$*

I give a qualitative proof of the conjecture. Let $E(p)$ be the excess demand as a continuous and differentiable function of price. Distinct and finite multiple equilibria imply that $\exists a, b$ such that

$$E(a) = E(b) = 0 : a, b > 0 \text{ and } a \neq b$$

By Rolle's theorem,

$$\exists c : E'(c) = 0 \text{ and } a < c < b$$

As a corollary, there is some $p*$ where $E'(p*) > 0$.

If total endowment is K then excess demand is total demand minus total endowment:

$$E(p_k) = \sum_{i=1}^{n} d_i(p_k) - K$$

$$\implies E'(p_k) = \sum_{i=1}^{n} d_i'(p_k)$$

Plugging in $p_k = p*$,

$$E'(p*) = \sum_{i=1}^{n} d_i'(p*) > 0 \implies \exists i : d_i'(p*) > 0$$

What happens to Germany and England's utilities at the different Walrasian equilibria? Another peculiarity:

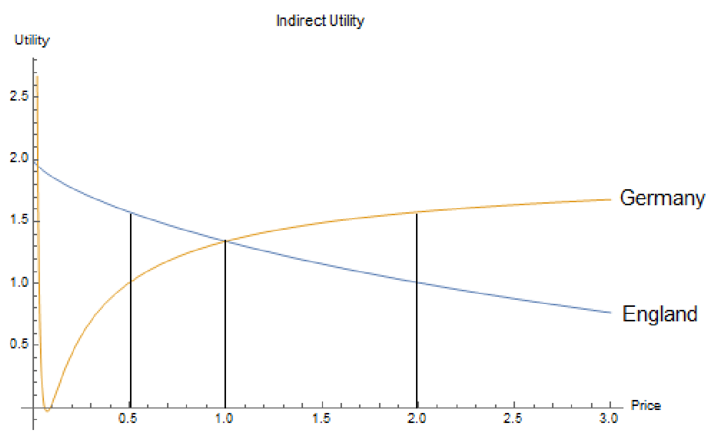


Figure 4: Indirect utility functions for Germany and England

Equilibrium Price	Germany	England
0.5	1.01	1.58
1	1.34	1.34
2	1.58	1.01

Table 2: Germany and England's utilities at the three Walrasian equilibrium prices

England has greater utility at $p = 0.5$, both have equal utility at $p = 1$ and Germany has greater utility when $p = 2$. Now, where will the outcome of the exchange economy lie? How can an allocation be assigned to an agent? There is no clear answer, at least based on the assumptions of the General equilibrium model. Suppose there was a "Walrasian auctioneer" who called out prices. England and Germany would call out how much Pound they would be willing to buy at every price. If, at a price, the demand for Pounds equalled the supply of Pounds, the auctioneer would determine that price and the resulting allocation as the Walrasian equilibrium.

Suppose the auctioneer started from a very high price. They would notice that the demand at the price is too low. Consequently, they would decrease the price in their next call. They would continue doing so, until they reach a price of 2, where excess demand would be zero. Germany would be left happier than England. If the auctioneer further decreased the price, demand would be (locally) greater than supply. Consequently, the auctioneer would drop the price in the next call. Hence, the price would eventually stabilize at 2.

Suppose the auctioneer started from price zero. At zero, demand would be higher than supply, so the auctioneer would start dropping the price. The price at which demand equalled supply would be 0.5. In this case, England would be happier than Germany. If the price rose further, supply would outstrip demand, causing the auctioneer to drop the price, stabilizing it at 0.5.

Such a mechanismm, called tatonnement, was proposed by Walras as an exchange mechanism in the General equilibrium model. In the case of multiple equilibria, we see that the exchange mechanism affects the final allocation in the exchange economy. This is not true of Cobb-Douglas preferences or of Leontief preferences. In the case of Cobb-Douglas preferences, there would be one price and allocation under tatonnement. In the case of Leontief preferences, there would be one price and many allocations under tatonnement. However, the agents would be indifferent under these allocations. The question of a potential welfare gain or loss due to transition to another Walrasian equilibrium would not arise.

Thus, multiple Walrasian equilibria seem to appear only under certain conditions. Furthermore, they seem to affect welfare outcomes. Samuelson (1947) found a weird implication of that fact: a transfer of wealth under certain conditions may *reverse* the welfare outcome.

The Transfer Paradox

Suppose England and Germany examined the Walrasian equilibria and decided to split the ground even and set p_s to 1. Now, suppose England asked Germany to pay war indemnities (this actually happened) and set the price at a modest 0.001 Pound (this did not happen). What happens when they go to trade with each other in general equilibrium?

Germany's endowment changed to $(2 - 0.001, r)$ and England's changed to $(r + 0.001, 2)$.

Graphing excess demand to find the Walrasian equilibrium prices,

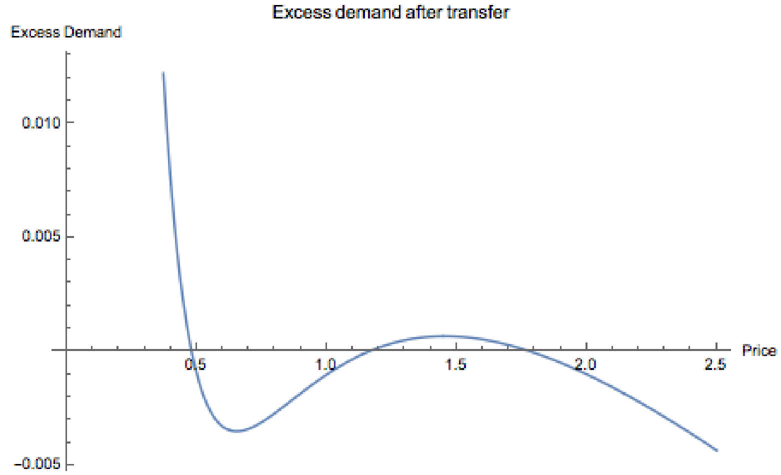


Figure 5: Excess Demand after the 0.001 transfer

The new equilibrium prices are 0.48, 1.16 and 1.78.

Comparing the conditions before and after the transfer,

Country	Equilibrium 1		Equilibrium 2		Equilibrium 3	
	p = 0.50	p = 0.48	p = 1.00	p = 1.16	P=2.00	P=1.78
Germany	1.01	0.98	**1.34**	**1.40**	1.58	1.54
England	1.58	1.59	**1.34**	**1.28**	1.01	1.07

Figure 6: Comparing the equilibria before and after the transfer

At the second equilibrium, Germany became strictly better off after the transfer and England strictly worse off. This is bizarre because Germany's wealth strictly fell and England's strictly increased. What happened?

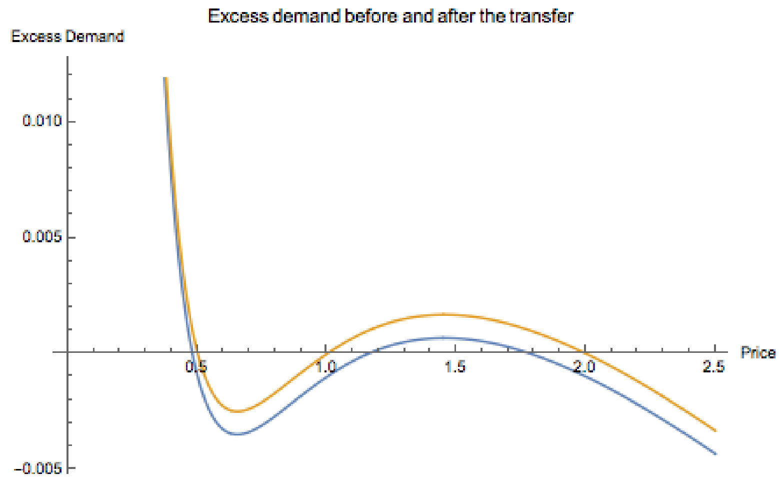


Figure 7: Excess demand before and after the transfer

The income effect on the excess demand decreased as Germany transferred some Pound to Britain. Consequently, the excess demand curve fell. But demand was upward sloping at price 1, a fall in demand corresponded to a *rise* in price given constant supply. Price increased, and it increased more than the fall in Germany's pound endowment, so it had a net positive income effect. As a result, Germany was left better off. On the other hand, as the pound was a consumption good for England, a rise in price of the Walrasian equilibrium made it worse off.

In a general equilibrium model, an agent's wealth is determined by its endowment and prices. But the endowment affects the price-determining mechanism. Consequently, there can be counter-acting effects on wealth. In this example, A proportional fall in endowment caused a more than proportional rise in price. Hence, there was a net positive effect on wealth.

This is the Transfer paradox, where a reallocation of endowments "immiserizes" the receiver while enriching the donor. The transfer paradox was first observed in this very debate surrounding war indemnities in 1929. It was Samuelson who identified this explanation behind the transfer paradox, connecting it with the presence of multiple equilibria.

Conclusion and Further Directions

Using England and Germany's example, I gleaned that the presence of multiple equilibria necessarily entails locally upward sloping demand for at least one agent. Furthermore, the Walrasian equilibrium outcome could not be determined by standard assumptions of general equilibrium theory because every agent strictly preferred a different equilibrium. The presence of such diverging outcomes could be used in constructing the transfer paradox where an endowment transfer leaves the donor better off and the receiver worse off.

These conclusions prompt more interesting enquiries.

First, what should be an allocation mechanism between multiple walrasian equilibria? In my example, every agent knows how to substitute one good for the other, but neither agent knows how to substitute one's utility for the other's. If the agents know their pay-offs at every equilibrium, perhaps a third party with its own preferences, the Walrasian auctioner, could set up a game to obtain a certain outcome. Perhaps the agents could play a simultaneous choice game, where each calls out the preferred equilibrium. A host of mechanism and allocation design problems can be posed from this point.

However, more closely, the England-Germany example was deliberately constructed. The utility functions were designed to be quasilinear, with a numeraire good and a consumption good. Furthermore, the endowments were designed to be small enough so that they produced a "swing" in the excess demand function. There are plenty more diverse examples in general equilibrium literature. However, these examples also follow specific constructions, and do not obviously suggest a generalization into a class of "multiple equilibria producing" problems. Hence, another useful pursuit could be to isolate the parameters that affect the presence of multiple equilibria in the exchange model.