Chebyshev and Sobolev Orthogonal Polynomials on the Sierpinski Gasket

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Introduction

We define a Sobolev inner product on the Sierpinski Gasket (SG) and discuss the properties of the corresponding orthogonal polynomials such as recurrence relations, finer estimates, and convergence with respect to parameters in the inner product. Furthermore, we define the Chebyshev polynomials on SG and find the first two Chebyshev polynomials in each monomial family.

Preliminaries

- Let $V_0 = \{q_0, q_1, q_2\} \in \mathbb{R}^2$ and $F_i(x) = \frac{1}{2}(x + q_i)$ for $i = 0, 1, 2$.

$$SG = \overline{\bigcup_{m=1}^{\infty} \bigcup_{|w|=m} F_w(V_0)}$$

Here $|w| = m \iff w \in \{0,1,2\}^m$ and $F_w = F_{i_m} \circ \cdots \circ F_{i_1}$ where $w = i_m \dots i_1$

- We work on the finite graph approximation $V_m = \bigcup_{|w|=m} F_w(V_0)$

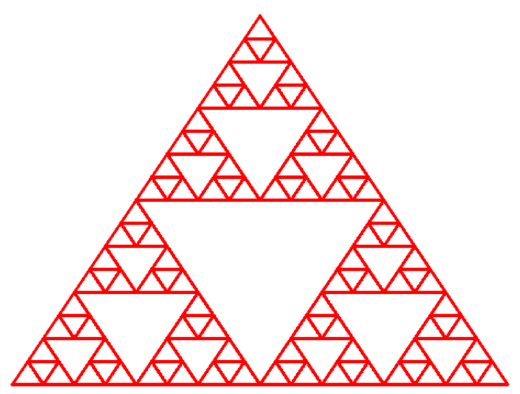

Figure 1: V_4

- Let $u : SG \mapsto \mathbb{R}$. Then the **Laplacian** Δ_μ is defined as

$$\Delta_\mu u(x) = \frac{3}{2} \lim_{m\to\infty} 5^m \Delta_m u(x)$$

- $G : SG \times SG \mapsto R$ is called **Green's Function** where

$$-\Delta u = f, u|_{V_0} = 0 \iff u(x) = \int_{SG} G(x,y) f(y)\, d\mu$$

- The space of polynomials with degree $\leq j$ is denoted $\mathcal{H}_j$
- We use the following basis $\{P_{jk}\}$ for $\mathcal{H}_m$ where $0 \leq j \leq m$ and $k = 1, 2, 3$:

$$\Delta^n P_{jk}(q_0) = \delta_{nj}\delta_{k1}$$
$$\Delta^n \partial_n P_{jk}(q_0) = \delta_{nj}\delta_{k2}$$
$$\Delta^n \partial_T P_{jk}(q_0) = \delta_{nj}\delta_{k3}$$

This is known as the **monomial basis**.

- Let f and g be polynomials on SG. Then the **Generalized Sobolev Inner Product** is defined as follows:

$$\langle f, g\rangle_{H^m} = \sum_{l=0}^{m} \lambda_l \int_{SG} \Delta^l f \Delta^l g\, d\mu + \sum_{l=0}^{m-1} \beta_l\, \varepsilon(\Delta^l f, \Delta^l g) +$$
$$\sum_{l=0}^{m-1} [\Delta^l f(q_0)\ \Delta^l f(q_1)\ \Delta^l f(q_2)] M_l [\Delta^l g(q_0)\ \Delta^l g(q_1)\ \Delta^l g(q_2)]^T$$

Here $\lambda_l, \beta_l > 0$ and M is a 3×3 symmetric positive definite matrix. For most of our results we use the H^1 inner product where $m = 1$ and $M = 0$.

- Fix $k = 1, 2$ or 3 and let $M_{nk} = \{f \mid f = \sum_{j=0}^{n} a_j P_{jk}, a_n = 1\}$. Then the **nth Chebyshev polynomial**, T_{nk} is defined to be the polynomial g such that

$$g := \min_{f \in M_{nk}} ||f||_\infty$$

Experimental Results and Applications

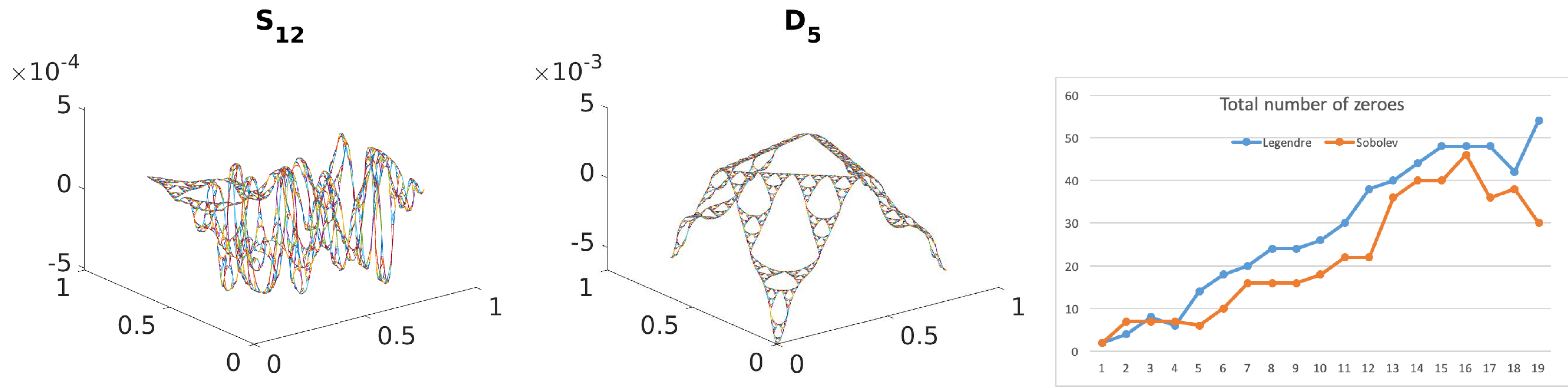

Figure 2: Degree 12 Anti-symmetric Sobolev polynomial, Degree 5 Symmetric Sobolev polynomial, Comparison between zeroes of Legendre and Sobolev polynomials on the edges

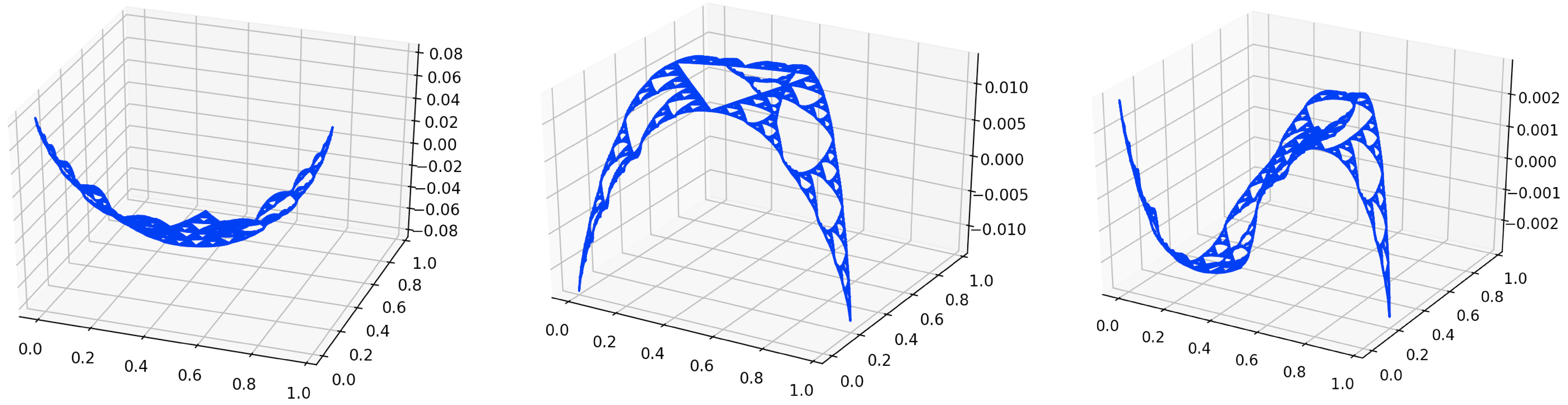

Figure 3: Left to right: First Chebyshev polynomials corresponding to $k = 1, 2, 3$

Theoretical results

Recurrence relations

Theorem. Suppose $k = 2$ or 3. For the Sobolev-m inner product (**??**), we have the following generalized recursion relation for $n \geq -1$

$$S_{n+m+1} - \mathcal{F}_{n+m+1} - \sum_{l=0}^{2m-1} a_{n,l} S_{n+m-l} = 0$$

where

$$a_{n,l} = -\frac{\langle \mathcal{F}_{n+m+1}, S_{n+m-l}\rangle_{H^m}}{\langle S_{n+m-l}, S_{n+m-l}\rangle_{H^m}}, \quad \mathcal{F}_{n+m+1} = \mathcal{G}^m p_{n+1}$$
$$\mathcal{G}(f)(x) := -\int_{SG} G(x,y) f(y) dy$$

and $S_j := 0$ if $j < 0$.

Theorem. Consider the H^1-inner product with $k = 1$ and let $\{S_n\}$ and $\{p_n\}$ be the corresonding monic Sobolev orthogonal Legendre polynomials respectively. Additionally, let $S_{-1} := 0$, $f_{n+2}(x) = -\int_{SG} G(x,y) p_{n+1}(y) dy$ and suppose that $\partial_n f_{n+2}(q_0) \neq 0$. Let $n \geq -1$. The Sobolev orthogonal polynomials satisfy the following recurrence relation:

$$S_{n+3} - a_n S_{n+2} - b_n S_{n+1} - c_n S_n = f_{n+3} + d_n f_{n+2}$$

The coefficients are given as follows:

$$a_n = -\frac{\langle f_{n+3} + d_n f_{n+2}, S_{n+2}\rangle_H}{\|S_{n+2}\|_H^2}$$
$$b_n = -\frac{\langle f_{n+3} + d_n f_{n+2}, S_{n+1}\rangle_H}{\|S_{n+1}\|_H^2}$$
$$d_n = -\frac{\partial_n f_{n+3}(q_0)}{\partial_n f_{n+2}(q_0)}, \ c_n = -d_n \frac{\|p_{n+1}\|_{L^2}^2}{\|S_n\|_H^2}$$

Convergence properties

Theorem. Suppose $k = 2$ or 3, and there exists $M > 0$ such that $\lambda_l \leq M$ for any $l < m$. Then there exists positive constants $C_1 = C_1(n, \mu)$, $C_2 = C_2(n, \mu, M, m)$ such that for any $n \geq 0$,

$$C_2 \geq \sum_{l=0}^{m-1} \lambda_l \int_{SG} (\Delta^l S_n)^2 d\mu \geq C_1$$

$$C_2 + \lambda_m \|p_{n-m}\|_{L^2}^2 \geq \|S_n\|_{H^m}^2 \geq C_1 + \lambda_m \|p_{n-m}\|_{L^2}^2$$

Consequently, for any $n \geq 2m + 1$, we have

$$\|S_n - \mathcal{F}_n\|_{L^2} \leq C(n, M, m, \mu)\lambda_m^{-1}$$

and $\lim_{\lambda_m \to \infty} \|\Delta^i S_n - \mathcal{G}^{m-i} p_{n-m}\|_{L^\infty} \to 0$ for any $0 \leq i \leq m$.

Theorem. Suppose the normal derivative conjecture is true and $k = 1$. Then there exists a sequence of monic polynomials $\{g_n\}_{n=0}^\infty$ independent of λ such that for any $n \geq 0$, $deg\, g_n = n$, S_n converges uniformly in x to g_n. And $g_{n+3} + d_n g_{n+2} = f_{n+3} + d_n f_{n+2}$ for any $n \geq 1$. For the basic cases, $g_0 = p_0$, $g_1 = p_1$, $g_2 + d_{-1} g_1 = f_2 + d_{-1} f_1 - \frac{\langle f_2 + d_{-1} f_1, g_0\rangle_{L^2}}{\|g_0\|_{L^2}^2} g_0$, and $g_3 + d_0 g_2 = f_3 + d_0 f_2 - \frac{\langle f_3 + d_0 f_2, g_0\rangle_{L^2}}{\|g_0\|_{L^2}^2} g_0$. Moreover, for any $\alpha < 1$, $n \geq 0$, $\lim_{\lambda \to \infty} \lambda^\alpha (S_n(\lambda) - g_n) = 0$ uniformly in x.

Further Research

1. **Interpolation and Quadrature**: For which points $x_1, \dots x_{3n+3}$ is the matrix M_n where

$$M_n = \begin{bmatrix} P_{0,1}(x_1) & \dots & P_{n,3}(x_1) \\ \vdots & \ddots & \vdots \\ P_{0,1}(x_{3n+3}) & \dots & P_{n,3}(x_{3n+3}) \end{bmatrix}$$

invertible? We call M_n the **interpolation matrix** and the above problem arises when we attempt to interpolate an n degree polynomial with $3n + 3$ distinct points. We have the following lemma:

Lemma 1. For any $n \geq 0$, take $x_i = F_0^{(i-1)}(q_1)$ for $1 \leq i \leq 2n + 2$, and $x_i = F_0^{(i-2n-3)}(q_2)$ for $2n + 3 \leq i \leq 3n + 3$. Then the matrix (**??**) is invertible.

On the other hand, S_{14} has at least 22 zeroes on the left half of SG so if we picked $x_1, \dots, x_{15}$ to be any 15 of these points then M_{14} would not be invertible. Thus the next question is to characterise those sets for which M_n is invertible. But due to the above lemma we have a quadrature rule I_n^m that integrates n-harmonic splines exactly. We have the following estimate for the error:

$$\left| I_n^m(f) - \int_{SG} f \right| \leq c_1(n) 5^{-(n+1)m} \|\Delta^{(n+1)} f\|_\infty$$

2. **Normal derivative conjecture**: Let $f_t = -\int_{SG} G(x,y) p_{t-1}\, dx$ where p_{t-1} is the $t - 1$th Legendre polynomial from the $k = 1$ family. Is $\partial_n f_t(q_0) \neq 0$? Values of $\partial_n f_t(q_0) \neq 0$ have been verified up to $t = 120$. We require this conjecture for the recurrence relation in the $k = 1$ case.
3. **Computing higher degree Chebyshev polynomials**: For the 2^{nd} Chebyshev polynomials of the families $k = 2$ and3, we can numerically find that $a_2 = 0.0619339$ and $a_3 = 0.0275013$. However, the general form of Chebyshev polynomials and their properties such as orthogonality, recursion, and alternation are still unexplored. [2], [3], [4], [6], [5], [1]

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